On an equation related to Stokes waves

Pichler-Tennenberg, Alex K.

Award date:
2002

Awarding institution:
University of Bath

On an Equation Related to Stokes Waves

submitted by

A.K. Pichler-Tennenberg

for the degree of Ph.D.

of the

University of Bath

2002

Summary

Local regularity theory, and local and global bifurcation theory is given for solutions of a generalisation of the Babenko equation for Stokes waves.

To my dear parents.

God demonstrates his own love for us in this:
While we were still sinners, Christ died for us.

Romans 8:5

Acknowledgements

Thanks go to the following people who have helped me in countless ways, and without whom neither I nor this thesis would have made it through:

John Toland, my supervisor, whose frequent deep insights are behind many of the good ideas in this work, and whose regular chiding ensured that I kept working most of the time.

Geoffrey Burton and Eugene Shargorodsky, my examiners, for reading the thesis (even Appendix C!) extremely thoroughly, and for pointing out numerous mistakes.

The E.P.S.R.C. for funding my studentship.

People with whom I have shared an office: Orlando Avila-Pozos, Adam Mawby, Claudia Carillo, Paul Spencer, Ramses Mena and Dan Crispin for their friendship, humour, help, patience and insights.

Maths Ph.D. types: Jeremy Penn, Rich Sharp and Pete Hartley for amusing "hours" spent over lunch and coffee; Kirill Cherednichenko, with whom I practically shared an office for a time, for knowing (and explaining) everything about everything.

The maths department computer support team, for providing an exceptionally reliable service, and for introducing me to Mrs C.; the departmental secretaries, for helping in all sorts of ways.

Mrs C., for providing the best tea-making service in the university, and for leaving me in no doubt about the completion of any transaction. 'Done'.

My father, for his great friendship, support and love over so many years; my grandmother for her prayers [about to be answered?] and for her last joke.

People with whom I have lived: Jon Heritage, Will Price, Mitch Piddock; Paul Riggs and Tim Taylor, for their ongoing fellowship, prayers and support; Ben Rotheray and Phil Bazlinton, for providing excellent company on Saturday mornings and on bike rides; Trevor, Sheila and David Dolby, for being everything good landlords - and good friends - should be.

Rob Bate, for leading me into cycling, photography and word invention, for providing constant hilaritude, for keeping a tatty picture of two Tirolean gentlemen on his wall, and for being a good friend.

My brothers and sisters in Christ: Jon Wright and Steve Bosanko, for their long-lasting and loyal friendship, and for many formative discussions; Ben Johnson, Graeme Ambler and John Mark Molyneux, for their rôle in my journey to faith; Steve WW, Martin King, Petra and Damien Harwin, for much support and prayer in troubling times; everyone in my homegroup, in B.U.C.U., and at Widcombe; Rupert Bentley-Taylor and Jonathan Skinner, whose preaching so often inspires me, challenges me and keeps me from falling; Dick Dowsett, for inadvertently suggesting what I should do next, and John Aitken, for helping me to organise it; Christy Pearson, for her love, patience and forgiveness.

Finally, Jesus, my creator, sustainer, saviour, lord and friend. A mathematician also.

Contents

Chapter 1

Introduction

1.1 Summary of results

This thesis is an investigation of solutions w in the "2π-periodic Sobolev space" $W^{1,1}_{2\pi}$ of the equation

$$f(w)(1+\mathcal{C}w')+\mathcal{C}(f(w)w')=c, \tag{1.1}$$

where $f \in C^0(J)$ for some $J \subset \mathbb{R}$, $\mathcal{C}$ is the "2π-periodic Hilbert Transform", defined on 2π-periodic real-valued Lebesgue-integrable functions, and c is a real constant.

Interest in (1.1) arises because the equation

$$\mathcal{C}w' = \lambda(w + w\mathcal{C}w' + \mathcal{C}(ww')), \tag{1.2}$$

given by taking $f(x) = 1 - 2\lambda x$ (where $\lambda \in \mathbb{R}$) and $c = 1$ in (1.1), is the Babenko equation (see [2]) satisfied by a "Stokes wave" of wavelength Λ, propagating with constant speed v in a uniform gravitational field g, where the "Froude number" is $1/\lambda = 2\pi v^2/g\Lambda$.

A Stokes wave is a steady periodic irrotational water wave on an infinitely deep incompressible fluid, with zero surface tension and viscosity in a uniform gravitational field. Physical considerations show that Stokes waves are described by solutions of the free boundary problem

$$\begin{array}{rcll}
\Delta\psi & = & 0 & \text{in } \Omega, \\
\psi(x,u(x)) & = & 0 & \text{for all } x \in \mathbb{R}, \\
\nabla\psi(x,y) & \to & (0,1) & \text{as } y \to -\infty, \\
\psi(x+2\pi,y) & = & \psi(x,y) & \text{for all } (x,y) \in \Omega, \\
\frac{1}{2}|\nabla\psi(x,u(x))|^2 + \lambda u(x) & \equiv & \frac{1}{2} & \text{for all } x \in \mathbb{R},
\end{array} \tag{1.3}$$

where the domain Ω (the region below the surface $\mathcal{S} = \{(x,u(x))|x \in \mathbb{R}\}$ of the wave) and the "stream function" ψ are unknown. The final equation in (1.3) expresses that atmospheric pressure is constant on $\mathcal{S}$, and has been shown [6] to be equivalent (after a conformal transformation) to the following condition

$$(1-2\lambda w)\{w'^2 + (1+\mathcal{C}w')^2\} = 1 \text{ almost everywhere,}$$

which gives rise to a Stokes wave with profile $\{(-(x+\mathcal{C}w(x)), w(x))|x \in \mathbb{R}\}$. We shall call this condition, along with the more general condition

$$f(w)\{w'^2 + (1+\mathcal{C}w')^2\} = c \text{ almost everywhere}$$

for solutions of (1.1), the **Bernoulli condition**. Each smooth solution of (1.2) such that $x \mapsto (-(x+\mathcal{C}w(x)), w(x))$ is injective on $\mathbb{R}$ satisfies the Bernoulli condition, and so describes a Stokes wave.

In [18] (see also [4], [17] and [19]) it is shown, using "Riemann-Hilbert theory", a bootstrapping argument and "Lewy's theorem" that solutions $w \in W^{1,1}_{2\pi}$ of (1.2) are real-analytic on $\{t \in [-\pi,\pi)|1-2\lambda w(t) \neq 0\}$. In Chapter 2 we generalise this result. We give local regularity results for solutions $w \in W^{1,1}_{2\pi}$ in the case where f is in the "Hölder class" $C^{k,\beta}$ on part of the range of w, and $\mathcal{C}w'$ is "locally Lebesgue integrable" on an open set. We also give a regularity result in the case where f is real-analytic on part of the range of w and w satisfies the same hypotheses. The conclusions found in the latter case are not as strong as those found in the above-mentioned papers unless f' is nowhere zero or w satisfies the Bernoulli condition.

Considerable work has been done ([17],[18] and [19]) to show that solutions

$w \in W^{1,1}_{2\pi}$ of (1.2) such that $1-2\lambda w(t) \geq 0$ almost everywhere satisfy the Bernoulli condition under very weak additional hypotheses. In Chapter 2 we also generalise some of this work. We give conditions for a solution w, in the "Hardy space" $\mathcal{H}^{1,1}_{2\pi}$, of (1.1) to satisfy the Bernoulli condition.

In [4] (see also [5] and [6]) it is shown, using the theory of "Crandall-Rabinowitz transversality", that for each $n \in \mathbb{N}$, a unique real-analytic curve of even nontrivial solutions $(\lambda, w) \in \mathbb{R} \times W^{1,2}_{2\pi}$ of (1.2) of "fundamental period" $2\pi/n$ "bifurcates" from the line $\mathbb{R} \times \{0\}$ of trivial solutions at $(n, 0)$ [note that $(0, k)$ is a solution for each constant function k]. Each such bifurcation is a "subcritical pitchfork bifurcation". In chapter 3 we generalise this result. We study the equation

$$f(\lambda, w)(1 + \mathcal{C}w') + \mathcal{C}(f(\lambda, w)w') = f(\lambda, 0), \tag{1.4}$$

where $f : I \times J \to \mathbb{R}$ (I and J open intervals in $\mathbb{R}$) [note that for all $\lambda \in I$, $(\lambda, 0)$ is a solution of (1.4)]. We find a *necessary* condition for $(\lambda^*, 0)$ to be a "bifurcation point", and three sets of *sufficient* conditions. The first of these sets of sufficient conditions, which uses the theory of "eigenvalue crossing numbers", is very general, but provides no information about the behaviour of the set of nontrivial solutions of (1.4) near to a bifurcation point.

The second set of conditions is a special case of the first and uses Crandall-Rabinowitz transversality. In this case we find that a unique *curve* of solutions bifurcates from $(\lambda^*, 0)$. Sufficient conditions are given in this case for solutions on the bifurcating curve to have fundamental period $2\pi/n$ or to be constant functions; and also for the curve to be a "supercritical or subcritical pitchfork" or to be "transcritical".

The third set of conditions is not included in the first. It is found by calculating the Taylor series of a function involved in the "bifurcation equation" of (1.4). In this case we find that two C^1 curves of nontrivial solutions of (1.4) to bifurcate from $I \times \{0\}$ at $(\lambda^*, 0)$.

In [5] (see also [6]) it is shown, using the "global analytic bifurcation theorem", that bifurcating curves of solutions of (1.2) may be continued globally. In Chapter 4 we generalise this. We give sufficient conditions on a real-analytic function f for a bifurcating curve of solutions of (1.4) to be continued globally. The global analytic bifurcation theorem gives three possibilities for the global continuation of an analytic curve of solutions of (1.4): either it continues until it reaches a point (λ, w) where $f(\lambda, w(t)) = 0$ for some $t \in S^1$, or it contains a sequence divergent in $\mathbb{R} \times W^{1,2}_{2\pi}$ to ∞, or it forms a closed loop. We give conditions sufficient to prevent the first two of these, and thereby ensure that the bifurcating curve forms a closed loop.

In the rest of this chapter, we introduce some well-known concepts and results which are used in the sequel.

1.2 Notation

The **interior** of a set E will be denoted E°, its **closure** $\overline{E}$ and its **boundary** ∂E. The **zero set** or **null space** of a function f will be denoted $\mathcal{N}(f)$ and its **range** $\mathcal{R}(f)$. If E is a subset of the domain of f then $f(E)$ will denote the **image** of E under f, and if $J \subset \mathcal{R}(f)$ then $f^{-1}(J)$ will denote the **inverse image** of J under f.

If $\phi \in C^k(U)$ where $U \subset \mathbb{R}^n$, then we write, for suitable $i_j \in \mathbb{N} \cup \{0\}$

$$\phi^{(i_1,\ldots,i_n)}(x_1,\ldots,x_n) = \frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}} \phi(x_1,\ldots,x_n).$$

We shall usually identify the interval $[-\pi, \pi)$ with the unit circle $S^1 \subset \mathbb{R}^2$, and identify 2π-periodic functions with maps on S^1. This is justified by the bijection $t \mapsto (\cos t, \sin t)$.

The **open unit disc** in $\mathbb{C}$ will be denoted $\mathcal{D}$. If $I \subset S^1$ $(= [-\pi, \pi))$ then we write $I^* = \{\mathrm{e}^{it} | t \in I\} \subset \partial\mathcal{D}$.

1.3 Function spaces, Hölder continuity, inequalities and the Faà-de Bruno formula

In this section we review some basic facts about function spaces (in particular L^p spaces and spaces of Hölder continuous functions), inequalities involving L^p functions and formulae for derivatives of composite functions.

Definition 1.1. For $p \in (0, \infty)$ and $I \subset \mathbb{R}$, the **Lebesgue space** $L^p(I)$ is the space of real-valued Lebesgue measurable functions u on I such that

$$\|u\|_{L^p(I)} = \left(\int_I |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < \infty.$$

The 2π**-periodic Lebesgue space** $L^p_{2\pi}$ is the space of real-valued 2π-periodic functions u such that $\|u\|_p := \|u\|_{L^p(S^1)} < \infty$.

$L^\infty(I)$ is the space of real-valued functions u on I such that

$$\|u\|_{L^\infty(I)} = \operatorname{ess\,sup}\{|u(x)|| x \in I\} < \infty;$$

and $L^\infty_{2\pi}$ is defined to be the space of real-valued 2π-periodic functions u such that $\|u\|_\infty := \|u\|_{L^\infty(S^1)} < \infty$.

The spaces $L^2(I)$ are Hilbert spaces with the inner product

$$\langle u, v \rangle_2 = \int_I u(x) v(x) \, \mathrm{d}x.$$

The **local Lebesgue space** $L^p_{\mathrm{loc}}(I)$ is the space of real-valued functions u on I such that $u \in L^p(\tilde{I})$ for each compact $\tilde{I} \subset I$. ■

On a bounded set I, if $p, q \in [1, \infty]$ with $p \leq q$ then $L^q(I) \subset L^p(I)$. In particular, $L^q_{2\pi} \subset L^p_{2\pi}$.

The following inequalities deal with functions in L^p spaces.

Hardy's inequality states that if $f \in L^p(0, \infty)$ $(1 < p < \infty)$ and $F : (0, \infty) \to \mathbb{R}$ is given by

$$F(x) = \frac{1}{x} \int_0^x f(t) \, \mathrm{d}t, \tag{1.5}$$

then $F \in L^p(0, \infty)$ also and $\|F\|_{L^p(0,\infty)} \leq \frac{p}{p-1} \|f\|_{L^p(0,\infty)}$. See [16] for a proof.

The **Hardy-Littlewood-Sobolev inequality** in one dimension states that if $f \in L^p(\mathbb{R})$ with $p > 1$, and $0 < \lambda < 1$ then the map $B : \mathbb{R} \to \mathbb{R}$ given by

$$B(x) = \int_{\mathbb{R}} f(y) |x - y|^{-\lambda} \, \mathrm{d}y \tag{1.6}$$

is in $L^r(\mathbb{R})$ (where $1/r = 1/p+\lambda-1$ if $1/p+\lambda-1 > 0$ and $r = \infty$ if $1/p+\lambda-1 < 0$) with $\|B\|_{L^r(\mathbb{R})} \leq C\|f\|_{L^p(\mathbb{R})}$ for some constant C, independent of f. See [14] for a proof.

Definition 1.2. The **Sobolev space** $W^{k,p}(I)$ (where $k \in \mathbb{N}$ and $p \in [1,\infty]$) is the space of real-valued functions on I which, along with their first k weak derivatives, are in $L^p(I)$. Each $W^{k,p}(I)$ is a Banach space under the norm

$$\|u\|_{W^{k,p}(I)} = \begin{cases} \left(\sum_{j=0}^{k} \|u^{(k)}\|_{L^p(I)}^p\right)^{\frac{1}{p}} & p < \infty \\ \max\{\|u^{(j)}\|_{L^\infty(I)} | 0 \leq j \leq k\} & p = \infty. \end{cases}$$

The spaces $W^{k,2}(I)$ are Hilbert spaces with the inner product

$$\langle u, v\rangle_{k,2} = \sum_{i=0}^{k} \int_I u^{(i)}(x)v^{(i)}(x) \ \mathrm{d}x.$$

The 2π**-periodic Sobolev spaces** $W_{2\pi}^{k,p}$ and the **local Sobolev spaces** $W_{\mathrm{loc}}^{k,p}(I)$ are defined analogously to the spaces $L_{2\pi}^p$ and $L_{\mathrm{loc}}^p(I)$ respectively. ■

A nonconstant function $u : \mathbb{R} \to \mathbb{R}$ has **fundamental period** $p > 0$ if u is p-periodic, and for all $0 < q < p$, u is not q-periodic.

We shall frequently have to take derivatives of compositions of functions. We give first a version of the **chain rule** for the composition of a C^1 function with a Sobolev function. If $w \in W^{1,p}(I)$ and $f \in C^1(\mathcal{R}(w))$, then $f \circ w$ is weakly differentiable and

$$(f \circ w)' = f'(w)w' \in L^p(I). \tag{1.7}$$

See [14] for a proof.

If E is an open interval in $\mathbb{R}$, $\psi \in C^n(E)$ and $\varphi \in C^n(\psi(E))$ then the nth derivative of $\varphi \circ \psi$ is given by the **Faà-de Bruno formula** [3]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\varphi(\psi(x)) = \sum^{n} \frac{n!}{\prod_{j=1}^{m} i_j!}\varphi^{(\sum_{j=1}^{m} i_j)}(\psi(x)) \prod_{j=1}^{m} \left(\frac{\psi^{(j)}(x)}{j!}\right)^{i_j}, \tag{1.8}$$

(where $\sum^{n}$ denotes the sum over non-negative integers $i_1, \ldots, i_m$ (for any $m \in \{0, \ldots, n\}$) satisfying $i_1 + \cdots + mi_m = n$). A more general version of this may be found in [6] and in [8].

Definition 1.3. Let I be an open subset of $\mathbb{R}^n$. We define the space $C^k(\overline{I})$ ($k \in \mathbb{N} \cup \{0\}$) to be the set of all functions $u \in C^k(I)$ such that all partial derivatives of u of order less than or equal to k are bounded and uniformly continuous on I.

If $u \in C^k(\overline{I})$ then all partial derivatives of u of order less than or equal to k have unique bounded continuous extensions to $\overline{I}$.

The set $C^{k,\alpha}(I)$ (where $k \in \mathbb{N} \cup \{0\}$ and $\alpha \in (0,1]$) comprises functions $u \in C^k(\overline{I})$ such that for all $x, y \in I$ and for any $m = (m_1, \ldots, m_n) \in (\mathbb{N} \cup \{0\})^n$ with $\sum_{j=1}^{n} m_j \leq k$,

$$|u^{(m_1,\ldots,m_n)}(x) - u^{(m_1,\ldots,m_n)}(y)| \leq C_m \|x - y|^\alpha \tag{1.9}$$

for some constants C_m, independent of x and y. We define also $C^{k,\alpha}(\overline{I}) = C^{k,\alpha}(I)$.

If $u \in C^{k,\alpha}(I)$ then we say that $u^{(k)}$ is **Hölder continuous with exponent** α.

Note that if u satisfies (1.9) on I then each partial derivative of u of order less than or equal to k has a unique C^0 extension to $\overline{I}$, and satisfies (1.9) on $\overline{I}$.

If I is bounded then $C^{0,\alpha}(I)$ is a Banach space under the norm

$$\|u\|_{C^{0,\alpha}(I)} = \|u\|_{L^\infty(\overline{I})} + \sup_{x,y \in I} \frac{|u(x) - u(y)|}{|x - y|^\alpha}. \tag{1.10}$$

The spaces $C^{k,\alpha}_{2\pi}$ and $C^{k,\alpha}_{\mathrm{loc}}(I)$ are defined analogously to the spaces $L^p_{2\pi}$ and $L^p_{\mathrm{loc}}(I)$, respectively.

On a bounded set I, if $\alpha, \beta \in (0,1]$ with $\alpha \geq \beta$ then $C^{0,\alpha}(I) \subset C^{0,\beta}(I)$. If I is convex and bounded (or is a finite union of convex bounded sets) then $C^1(\overline{I}) \subset C^{0,1}(\overline{I})$ (proof via the mean-value theorem); otherwise we have only that $C^1(\bar{I}) \subset C^{0,1}_{\text{loc}}(I)$. ■

If $p > 1$, then $W^{1,p}_{2\pi}$ is compactly embedded in $C^{0,\beta}_{2\pi} \subset L^\infty_{2\pi}$ for each $\beta < 1-1/p$ (A subspace X of a Banach space Y is **compactly embedded** in Y if the map $\iota : X \to Y$ given by $\iota(x) = x$ for all $x \in X$ is compact; see Section 1.5), and is continuously embedded in $C^{0,1-1/p}_{2\pi}$.

If $w \in W^{1,p}_{2\pi}$, a simple argument using Hölder's inequality and the periodicity of w shows that the **amplitude** of w, $\text{Amp}(w) = \sup_{x,y \in S^1} |w(x) - w(y)|$ satisfies

$$\text{Amp}(w) \leq \frac{(2\pi)^{1-1/p}}{2} \|w'\|_p, \tag{1.11}$$

so that $\|w\|_\infty \leq (2\pi)^{1-1/p} \|w\|_{W^{1,p}_{2\pi}}$.

The following theorem deals with the Hölder continuity of products, quotients and compositions of Hölder continuous functions on bounded sets in $\mathbb{R}$. It is easily proved using the Faà-de Bruno formula (1.8), Leibniz' rule and the above definition of $C^{k,\alpha}(I)$.

Theorem 1.4. *Let $n \in \mathbb{N} \cup \{0\}$, $\alpha, \beta \in (0,1]$ and let $E \subset \mathbb{R}$ be bounded. Suppose $\varphi \in C^{n,\alpha}(E)$ and $\psi \in C^{n,\beta}(E)$. Then*

(i) $\varphi\psi \in C^{n,\min\{\alpha,\beta\}}(E)$.

(ii) If ψ is bounded away from 0 then $\varphi/\psi \in C^{n,\min\{\alpha,\beta\}}(E)$.

Suppose now that $\psi \in C^{n,\alpha}(E)$ and $\varphi \in C^{n,\beta}(\psi(E))$.

(iii) If $n = 0$ then $\varphi(\psi) \in C^{0,\alpha\beta}(E)$.

(iv) If $n \in \mathbb{N}$ then $\varphi(\psi) \in C^{n,\min\{\alpha,\beta\}}(E)$.

The following lemma deals with the Hölder continuity *up to the boundary* of a solution $u \in C^2(\Omega)$ (Ω a domain in $\mathbb{R}^n$) of a strictly elliptic equation $Lu = f$ (a second order differential operator

$$L = \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) D_{ij} + \sum_{i=1}^n b_i(x) D_i + c(x) \tag{1.12}$$

is **strictly elliptic** on Ω if the eigenvalues of the matrices $(a_{ij}(x))$ are strictly positive and uniformly bounded away from 0 on Ω).

Lemma 1.5. *Let Ω be a domain in $\mathbb{R}^n$ with a boundary portion T which is an open subset of an $n-1$-dimensional hyperplane in $\mathbb{R}^n$; let L, given by (1.12) be strictly elliptic on Ω; and suppose that $a_{ij}, b_i, c, f \in C^{0,\alpha}(\overline{\Omega})$ for all i and j. Suppose $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies $Lu = f$ and is such that on T, $u \equiv \phi$, where $\phi \in C^{2,\alpha}(\overline{\Omega})$. Then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega \cup T)$.*

A proof is to be found (with much weaker hypotheses on T) in [9].

1.4 Complex analysis: the Hilbert transform and Hardy spaces

In this section we review some standard and some less well-known facts about complex analysis, which will be used in Chapter 2.

If $U : \mathcal{D} \to \mathbb{C}$ then we define, for $r \in (0,1)$, $U_r : S^1 \to \mathbb{R}$ by $U_r(t) = U(r\mathrm{e}^{it})$. Where the limit exists we define also $U^*(t) = \lim_{r \to 1} U_r(t)$.

Definition 1.6. The 2π**-periodic Hilbert transform** $\mathcal{C}w$ of a function $w \in L^1_{2\pi}$ is defined as follows. The function $F : \mathcal{D} \to \mathbb{C}$ given by

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{it} + z}{\mathrm{e}^{it} - z} w(t)\ \mathrm{d}t \tag{1.13}$$

is holomorphic on $\mathcal{D}$ (see [16] §11.6). Let $F = U + iV$ [U is the Poisson Integral of w]. Then $F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} w(t)\ \mathrm{d}t$. For almost every $t \in S^1$, $U^*(t)$ and $V^*(t)$ exist (see [12]), and $U^*(t) = w(t)$. $\mathcal{C}w(t)$ is defined, for almost every t by

$$\mathcal{C}w(t) = V^*(t) \tag{1.14}$$

[note that $V(0) = 0$, so $\int_{-\pi}^{\pi} \mathcal{C}w(t)\ \mathrm{d}t = 0$]. In practice, $\mathcal{C}w(t)$ is found, for almost every t, either from the Cauchy principal value integral

$$\mathcal{C}w(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{w(y)}{\tan\frac{t-y}{2}}\,\mathrm{d}y,$$

a simple consequence of (1.13-1.14), or (in the case $w \in L^2_{2\pi}$) by linearity and a density argument on the Fourier coefficients of w: for $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{C}(\sin nt) = -\cos nt, \quad \mathcal{C}(\cos mt) = \sin mt.$$

■

The theorems of M.Riesz and of Privalov give information about the Hilbert transform of $L^p_{2\pi}$ ($p > 1$) functions and $C^{0,\alpha}$ ($\alpha < 1$) functions respectively.

Theorem 1.7. *(M. Riesz, Privalov)*

(A) $\mathcal{C}$ is a bounded linear operator from $L^p_{2\pi}$ into itself for all $p \in (1,\infty)$. For $u \in L^2_{2\pi}$, $\|\mathcal{C}u\|_2 \leq \|u\|_2$.

(B) $\mathcal{C}$ is a bounded linear operator from $C^{0,\alpha}_{2\pi}$ into itself for all $\alpha \in (0,1)$.

The local versions of the theorems of M.Riesz and Privalov given below, will be of great importance in Chapter 2. Proofs are to be found in Appendix A.

Theorem 1.8. *Let $u \in L^1_{2\pi}$ and $E \subset S^1$*

(1) If $u \in L^p(E)$ where $p \in (1,\infty)$ and $E \subset S^1$, then $\mathcal{C}u \in L^p_{\mathrm{loc}}(E^\circ)$.

(2) If $u \in C^{n,\alpha}(E)$, where $\alpha \in (0,1)$ and $n \in \mathbb{N} \cup \{0\}$, then $\mathcal{C}u \in C^{n,\alpha}_{\mathrm{loc}}(E^\circ)$.

Neither of these theorems provides any information about the 2π-periodic Hilbert transform of a general $L^1_{2\pi}$-function: indeed $\mathcal{C}$ does not map $L^1_{2\pi}$ into itself. However

Theorem 1.9. *(Kolmogorov) If $u \in L^1_{2\pi}$ then $\mathcal{C}u \in \cap_{p<1} L^p_{2\pi}$.*

See [12], chapter V, section C, theorem 5° for a proof.

Definition 1.10. The **Real Hardy Space** $\mathcal{H}^1_{\mathbb{R}}$ is the set of functions $u \in L^1_{2\pi}$ such that $\mathcal{C}u \in L^1_{2\pi}$. It is a Banach space under the norm $\|u\|_{\mathcal{H}^1_{\mathbb{R}}} = \|u\|_1 + \|\mathcal{C}u\|_1$. $\mathcal{H}^{1,1}_{\mathbb{R}}$ is the set of functions $u \in \mathcal{H}^1_{\mathbb{R}}$ such that $u' \in \mathcal{H}^1_{\mathbb{R}}$. It is a Banach space under the norm $\|u\|_{\mathcal{H}^{1,1}_{\mathbb{R}}} = \|u\|_{\mathcal{H}^1_{\mathbb{R}}} + \|u'\|_{\mathcal{H}^1_{\mathbb{R}}}$. ■

If $u \in L^1_{2\pi}$ and $\mathcal{C}u \geq f \in L^1_{2\pi}$ then $u \in \mathcal{H}^1_{\mathbb{R}}$. See [19] for a proof.

The following properties of $\mathcal{H}^1_{\mathbb{R}}$ and $\mathcal{H}^{1,1}_{\mathbb{R}}$ may be found in [22]. If $u, v \in \mathcal{H}^{1,1}_{\mathbb{R}}$, then

$$\int_{-\pi}^{\pi} u(x)\mathcal{C}v'(x)\ \mathrm{d}x = \int_{-\pi}^{\pi} v(x)\mathcal{C}u'(x)\ \mathrm{d}x. \tag{1.15}$$

If $u \in \mathcal{H}^1_{\mathbb{R}}$ then $u(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(y)\ \mathrm{d}y - \mathcal{C}^2 u(x)$.

Definition 1.11. For $p \in (0, \infty]$, the **Hardy Class** $\mathcal{H}^p_{\mathbb{C}}$ is defined to be the set of all holomorphic functions on U on $\mathcal{D}$ such that

$$\|U\|_{\mathcal{H}^p_{\mathbb{C}}} = \lim_{r\to 1} |||U_r|||_p = \sup_{r\in(0,1)} |||U_r|||_p < \infty,$$

■

The space Log comprises functions $u : S^1 \to \mathbb{C}$ such that $\log|u| \in L^1_{2\pi}$. If $u \in \text{Log}$ then $u \neq 0$ almost everywhere.

$\mathcal{H}^p_{\mathbb{C}}$ has the following properties:

(i) For any $F \in \mathcal{H}^p_{\mathbb{C}}$, $F^*(t)$ is well defined for almost every $t \in S^1$. Also $|F^*| \in L^p_{2\pi}$ with $|||F^*|||_p = \|F\|_{\mathcal{H}^p_{\mathbb{C}}}$; and if $F \neq 0$ then $F^* \in \text{Log}$.

(ii) If $F \in \mathcal{H}^1_{\mathbb{C}}$, then F is the Cauchy integral and the Poisson integral of F^* on S^1.

Further, we have

Theorem 1.12. *(Smirnov) If $F \in \mathcal{H}^p_{\mathbb{C}}$ for some $p > 0$ and $|F^*| \in L^q_{2\pi}$ for some $q > p$ then $F \in \mathcal{H}^q_{\mathbb{C}}$.*

Note the following consequence of the definitions of $\mathcal{H}^p_{\mathbb{C}}$ (Definition 1.11) and $\mathcal{C}$ (Definition 1.6): $|u + i\mathcal{C}u| \in L^p_{2\pi}$ ($p \in (0, \infty]$) if and only if $u + i\mathcal{C}u = F^*$, where

$F \in \mathcal{H}^p_{\mathbb{C}}$ and $F(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\ dt$. This observation, together together with the Theorems of M. Riesz (Theorem 1.7 *(A)*) and Kolmogorov (Theorem 1.10) leads to the following further properties of $\mathcal{H}^p_{\mathbb{C}}$.

(iii) $u \in L^1_{2\pi}$ if and only if $u + iCu = F^*$ for some $F \in \cap_{p<1} H^p_{\mathbb{C}}$ with $F(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\ dt$.

(iv) $u \in \mathcal{H}^1_{\mathbb{R}}$ if and only if $u + iCu = F^*$ for some $H \in \mathcal{H}^1_{\mathbb{C}}$ with $F(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\ dt$.

(v) $u \in L^p_{2\pi}$ $(p \in (0,\infty))$ if and only if $u + iCu = F^*$ for some $F \in H^p_{\mathbb{C}}$ with $F(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(t)\ dt$.

For $u \in \text{Log}$, an **outer function** $\mathcal{O}(u) : \mathcal{D} \to \mathbb{C}$ is defined by

$$\mathcal{O}(u)(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{iy}+z}{e^{iy}-z} \log|u(y)|\ dy\right\}. \tag{1.16}$$

(1.13) and (1.16) give that the boundary data of $\mathcal{O}(u)$ is given by

$$(\mathcal{O}(u))^*(x) = |u(x)|e^{iC(\log|u|)(x)} \tag{1.17}$$

Outer functions have the following properties:

(a) $\mathcal{O}(u)$ is holomorphic and has no zeros on $\mathcal{D}$.

(b) For $u_1, u_2 \in \text{Log}$, $\mathcal{O}(u_1u_2) = \mathcal{O}(u_1)\mathcal{O}(u_2)$.

(c) If $u \in \text{Log}$ and $p \in (0,\infty]$, then $|u| \in L^p_{2\pi}$ if and only if $\mathcal{O}(u) \in \mathcal{H}^p_{\mathbb{C}}$.

(d) For $U \in \mathcal{H}^p_{\mathbb{C}}$ $(p \in (0,\infty])$, $|U(z)| \le |\mathcal{O}(U^*)(z)|$ for all $z \in \mathcal{D}$.

The following theorem of Lewy ([13], [20]) is an important extension theorem for holomorphic functions on a half disc.

Theorem 1.13. *(Lewy) Suppose that* $F = U + iV$ *is holomorphic on* $D_r = \{x + iy \in \mathbb{C} | x^2 + y^2 < r, y < 0\}$ *and* $U, V \in C^1(\overline{D_r})$. *Suppose also that for* $|x| < r$,

$$U_y(x,0) = \mathcal{A}(x, U(x,0), V(x,0), U_x(x,0)),$$

where $\mathcal{A}$ is a complex-valued analytic function of all its arguments in a neighbourhood of $(0, U(0,0), V(0,0), U_x(0,0)) \in \mathbb{C}^4$*, which is real-valued when its arguments are.*

Then there exists a disc $\tilde{D}$*, centred at* 0*, and a holomorphic function* $\tilde{U} + i\tilde{V} : \tilde{D} \to \mathbb{C}$ *such that on* $D_r \cap \tilde{D}$*,*

$$U + iV = \tilde{U} + i\tilde{V}.$$

1.5 Operators, Fréchet differentiation and analyticity

In this section we review some facts about operators, Fréchet differentiability and analyticity, which are needed mainly in Chapters 3 and 4. We follow [6].

Let X and Y be Banach spaces over $\mathbb{R}$, and U an open subset of X.

Definition 1.14. A map $F : U \to Y$ is said to be **Fréchet differentiable** at $\xi \in U$ if and only if there exists $A \in \mathcal{L}(X, Y)$ such that

$$\lim_{\|h\|_X \to 0} \frac{\|F(\xi + h) - F(\xi) - Ah\|_Y}{\|h\|_X} = 0.$$

In this case we say that A is the **Fréchet derivative of F at ξ** and write, for $h \in X$, $Ah = \mathrm{d}F(\xi)h$.

If $\mathrm{d}F$ is Fréchet differentiable at $\xi \in U$ then we write, $\mathrm{d}(\mathrm{d}F)(\xi) = \mathrm{d}^2F(\xi)$. $\mathrm{d}^2F(\xi)$ is called the **second Fréchet derivative of F at ξ**. In the same way we may define, for suitable F and for $k \in \mathbb{N}$, the **kth Fréchet derivative of F at ξ**, $\mathrm{d}^kF(\xi)$.

d^kF is a map from U into $\mathcal{L}(X, \mathcal{L}(X, \dots, \mathcal{L}(X, Y)\dots))$, which may be identified with $\mathcal{L}^k(X, Y)$, the set of symmetric, k-linear forms on X. We write the image of $(x_1, \dots, x_n) \in X^n$ under $\mathrm{d}^kF(\xi)$ as $\mathrm{d}^kF(\xi)[x_1, \dots, x_n]$. If d^kF is continuous on U then we say that F is k-times continuously [Fréchet] differentiable on

U and write $F \in C^k(U,Y)$.

If F is a suitable function from an open subset U of $X_1 \times \cdots \times X_n$ into Y, where $X_1, \ldots, X_n$ are Banach spaces over $\mathbb{R}$, then we may form the **partial Fréchet derivatives** $\partial^{1,0,\ldots,0}F(\xi_1,\ldots,\xi_n), \ldots, \partial^{0,\ldots,0,1}F(\xi_1,\ldots,\xi_n)$ and higher order derivatives. $F \in C^k(U,Y)$ if and only if each of its partial derivatives of order less than or equal to k is continuous; if $F \in C^1(U,Y)$ then

$$\mathrm{d}F(\xi_1,\ldots,\xi_n)[(x_1,\ldots,x_n)] = \sum_{i=1}^{n} \partial^{0,\ldots,0,1,0,\ldots,0}F(\xi_1,\ldots,\xi_n)x_i.$$

If F is k-times differentiable at ξ then the order in which the mixed partial derivatives are taken is inconsequential. In this case we write

$$\partial^{i_1,0,\ldots,0} \cdots \partial^{0,\ldots,0,i_n}F(\xi_1,\ldots,\xi_n) = \partial^{i_1,\ldots,i_n}F(\xi_1,\ldots,\xi_n).$$

See [6] §3 for further details. ■

Definition 1.15. A map $F : U \to Y$ is **compact** if for every bounded sequence (x_n) in U, $(F(x_n))$ has a convergent subsequence.

A bounded linear operator $A \in \mathcal{L}(X,Y)$ is **Fredholm** if $\dim(\mathcal{N}(A))$ and $\operatorname{codim}(\mathcal{R}(A))$ are finite. The **index** of A is then $\dim(\mathcal{N}(A)) - \operatorname{codim}(\mathcal{R}(A))$. ■

Clearly any linear homeomorphism is Fredholm with index 0. More generally, if $A \in \mathcal{L}(X,Y)$ can be written as the sum of a compact linear operator and a linear homeomorphism then A is Fredholm with index 0.

If F is compact from U to Y then for $x \in U$, $\mathrm{d}F(x)$ is compact from X to Y.

Definition 1.16. If $F \in C^n(U,Y)$ $(n \in \mathbb{N} \cup \{0\})$ and $\xi, x \in U$ then

$$F(\xi) - \sum_{k=0}^{n} \frac{1}{k!}\mathrm{d}^k F(x)[\xi - x, \ldots, \xi - x] = R_n(\xi, x), \tag{1.18}$$

where $R_n(\xi,x)$ is the nth order **Taylor series remainder** of F about x at ξ.

If $F \in C^\infty(U,Y)$ then, whether it converges or not, the **formal Taylor series** of F about $x \in U$ and at $\xi \in U$ is

$$\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{d}^k F(x)[\xi - x, \ldots, \xi - x].$$

■

Taylor's theorem states that if $F \in C^{n+1}(U,Y)$ then

$$\|R_n(\xi,x)\|_Y \leq \frac{\|\xi - x\|_X^{n+1}}{(n+1)!} \sup_{0 \leq t \leq 1} \|\mathrm{d}^{n+1} F((1-t)x + t\xi)\|_{\mathcal{L}^{n+1}(X,Y)}. \tag{1.19}$$

If $X = \mathbb{R}$ and $f \in C^n(U,\mathbb{R})$ then we have the following result (see [7]):

$$R_n(\xi,x) = \frac{(\xi - x)^n}{(n-1)!} \int_0^1 (1-t)^{n-1} [f^{(n)}((1-t)x + t\xi) - f^{(n)}(x)] \,\mathrm{d}t. \tag{1.20}$$

Definition 1.17. F is **real-analytic** at x if, for all $\xi \in U$ with $\|\xi - x\|_X$ sufficiently small,

$$F(\xi) = \sum_{k=0}^{\infty} m_k[\xi - x, \ldots, \xi - x], \tag{1.21}$$

for some $m_k \in \mathcal{L}^k(X,Y)$ such that

$$\sup_{k \in \mathbb{N} \cup \{0\}} r^k \|m_k\|_{\mathcal{L}^k(X,Y)} = M < \infty \tag{1.22}$$

for some $r > 0$. ■

If F is defined by (1.21) and (1.22) holds then F is real-analytic at each point of $U_x = \{\xi \in U | \|\xi - x\|_X < r\}$. We have that $F \in C^\infty(U_x, Y)$ with

$$m_k = \frac{1}{k!} \mathrm{d}^k F(x).$$

Also, for $k \in \mathbb{N}$, $\mathrm{d}^k F$ is analytic on U_x, and for some constants $C > 1$ and $R < 1$,

$$\|\mathrm{d}^k F(\xi)\|_{\mathcal{L}^k(X,Y)} \leq C\frac{k!}{R^k} \tag{1.23}$$

for all $k \in \mathbb{N} \cup \{0\}$ and all $\xi \in U$ such that $\|\xi - x\|_X < r/2$.

If F is real-analytic on U and $G : F(U) \to Z$ (Z a Banach space) is real-analytic, then $F \circ G$ is real-analytic on $F(U)$.

Let $V \subset Y$ be open. The **chain rule** states that if $F : U \to V$ and $G : V \to Z$ are such that $\mathrm{d}F(x)$ and $\mathrm{d}G(F(x))$ exist then $\mathrm{d}(G \circ F)(x)$ exists and

$$\mathrm{d}(G \circ F)(\xi)h = \mathrm{d}G(F(\xi))[\mathrm{d}F(\xi)h].$$

If X is a Hilbert space, then a functional $\varphi \in C^2(U)$, where U is an open neighbourhood of $\xi \in X$, is said to have a **nondegenerate critical point** at ξ if $\mathrm{d}\varphi(\xi) = 0$ and $\mathrm{d}^2\varphi(\xi)[u,v] = \langle Lu, v\rangle_X$ for some invertible linear operator $L : X \to X$. In this case we have the following theorem, which will be used at the end of Chapter 3. See [10] and [15] for a proof.

Theorem 1.18. *(Morse lemma) Let X be a Hilbert space, and suppose $\varphi \in C^k(U)$ (where U is an open neighbourhood of 0 and $k \geq 3$) has a nondegenerate critical point at 0. Then there exists an open neighbourhood A of 0 and a local C^{k-2} diffeomorphism (a C^{k-2}-function with C^{k-2}-inverse) $\Gamma : A \to U$ such that $\Gamma(0) = 0$ and*

$$\varphi(\Gamma(u)) = \varphi(0) + \frac{1}{2}\mathrm{d}^2\varphi(0)[u,u].$$

If φ is real-analytic, then so too is Γ.

1.6 Local bifurcation theory

Bifurcation theory deals with the structure of the solution set of an equation

$$F(\lambda, x) = 0 \tag{1.24}$$

where $F : U \to Y$ (U an open neighbourhood of a point $(\lambda^*, 0)$ in $\mathbb{R} \times X$) is such that $F(\lambda, 0) = 0$ for all λ. We seek to find points $(\lambda^*, 0) \in \mathbb{R} \times X$ at which nontrivial solutions (λ, x) "bifurcate" from the line $\{(\lambda, 0) | \lambda \in \mathbb{R}\}$ of trivial solutions. We follow [1] and [6].

Definition 1.19. $(\lambda^*, 0)$ is a **bifurcation point** on the line of trivial solutions of (1.24) if and only if it is in the closure of $\{(\lambda, x) \in \mathbb{R} \times (X \setminus \{0\}) | F(\lambda, x) = 0\}$ in $\mathbb{R} \times X$. ■

Using the implicit function theorem, it is easy to find a necessary condition for a point $(\lambda^*, 0)$ to be a bifurcation point:

Lemma 1.20. *(a necessary condition) Suppose $F \in C^1(U, Y)$. If $\partial^{0,1} F(\lambda^*, 0)$ is a homeomorphism then $(\lambda^*, 0)$ is not a bifurcation point of (1.24).*

The proof is a simple application of the implicit function theorem.

The following theorem gives a sufficient condition for a point $(\lambda^*, 0)$ to be a bifurcation point of (1.24) in the case $X \subset Y$. It uses the notion of a crossing number, defined as follows.

Suppose that for some $\lambda^* \in \mathbb{R}$ such that $(\lambda^*, 0) \in U$, the linear operator $\partial^{0,1} F(\lambda^*, 0) : X \to Y$ has an isolated eigenvalue 0 with finite [algebraic] multiplicity. Let, for λ in a neighbourhood of λ^*, $\sigma(\lambda)$ be the sum of the algebraic multiplicities of the negative eigenvalues of $\partial^{0,1} F(\lambda, 0)$ which converge to 0 as $\lambda \to \lambda^*$ (this is finite since all eigenvalues of $\partial^{0,1} F(\lambda, 0)$ which converge to 0 as $\lambda \to \lambda^*$ are perturbations of the eigenvalue 0 of $\partial^{0,1} F(\lambda^*, 0)$). The **crossing number**, if it exists, of $\partial^{0,1} F(\lambda, 0)$ at λ^* is then given by

$$\chi(\partial^{0,1} F(\lambda, 0), \lambda^*) = \lim_{\mu \searrow 0} \sigma(\lambda^* - \mu) - \sigma(\lambda^* + \mu).$$

Theorem 1.21. *Suppose $X \subset Y$, and $F \in C^1(U, Y)$, where U is a neighbourhood of $(\lambda^*, 0) \in \mathbb{R} \times X$, is such that $F(\lambda, 0) = 0$ for all λ. Suppose also that $\partial^{0,1} F(\lambda^*, 0)$ is a Fredholm operator with index 0, and has an isolated eigenvalue 0 of finite algebraic multiplicity $n \in \mathbb{N}$ (so is not a homeomorphism).*

If $\chi(\partial^{0,1} F(\lambda, 0), \lambda^*)$ exists and is odd, then $(\lambda^*, 0)$ is a bifurcation point of (1.24).

A proof is to be found in [11].

If F is sufficiently regular at $(\lambda^*, 0)$, and $\partial^{0,1}F(\lambda^*, 0)$ is Fredholm with index 0 and has one-dimensional null space, then the following condition on $\partial^{1,1}F(\lambda^*, 0)$ is sufficient to show that a *curve* of nontrivial solutions bifurcates from the line of trivial solutions at $(\lambda^*, 0)$.

Theorem 1.22. *(Crandall-Rabinowitz transversality) Suppose $F \in C^k(U, Y)$ $(k \geq 2)$ is such that $\partial^{0,1}F(\lambda^*, 0)$ is a Fredholm operator with index 0, with $\mathcal{N}(\partial^{0,1}F(\lambda^*, 0)) = \operatorname{span}\{v^*\}$ for some $v^* \in X \setminus \{0\}$. If the* ***transversality condition,***

$$\partial^{1,1}F(\lambda^*, 0)[1, v^*] \notin \mathcal{R}(\partial^{0,1}F(\lambda^*, 0)),$$

holds then a unique C^{k-1} curve $\mathfrak{B}$ of solutions of (1.24) bifurcates from $(\lambda^*, 0)$. If F is real-analytic, then so too is $\mathfrak{B}$.

The first step in the proof of Theorem 1.22 is the reduction of the problem to a finite-dimensional one. This is achieved via the Lyapunov-Schmidt reduction, which applies in a wider setting than that of Theorem 1.22:

Lemma 1.23. *(Lyapunov-Schmidt reduction) Suppose $F \in C^k(U, Y)$ is such that $\partial^{0,1}F(\lambda^*, 0)$ is a Fredholm operator with $\mathcal{N}(\partial^{0,1}F(\lambda^*, 0)) \neq \{0\}$ (so $\partial^{0,1}F(\lambda^*, 0)$ is not a homeomorphism). Suppose also that $X = \mathcal{N}(\partial^{0,1}F(\lambda^*, 0)) \oplus W$ and $Y = \mathcal{R}(\partial^{0,1}F(\lambda^*, 0)) \oplus Z$.*

We write, for $x \in X$, $x = x_N + x_W$, where $x_N \in \mathcal{N}(\partial^{0,1}F(\lambda^*, 0))$ and $x_W \in W$. If P is the projection operator from Y onto Z with $\mathcal{N}(P) = \mathcal{R}(\partial^{0,1}F(\lambda^*, 0))$, then there exist neighbourhoods Λ of λ^* in $\mathbb{R}$, $\mathcal{N}$ of 0 in $\mathcal{N}(\partial^{0,1}F(\lambda^*, 0))$ and $\mathcal{W}$ of 0 in W, and a map $\gamma \in C^k(\Lambda \times \mathcal{N}, \mathcal{W})$ such that in $\Lambda \times (\mathcal{N} \oplus \mathcal{W})$, $F(\lambda, x) = 0$ if and only if $x_W = \gamma(\lambda, x_N)$ and the ***bifurcation equation***

$$PF(\lambda, x_N + \gamma(\lambda, x_N)) = 0 \tag{1.25}$$

holds. The map γ satisfies $\gamma(\lambda, 0) = 0$ for all λ, and $\partial^{0,1}\gamma(\lambda^*, 0) = 0$. If F is real-analytic then so too is γ.

There follows a sketch of the remainder of the proof of Theorem 1.22.

If $\partial^{0,1}F(\lambda^*,0)$ satisfies Theorem 1.22 then $\operatorname{codim}(\mathcal{R}(\partial^{0,1}F(\lambda^*,0))) = 1$ and for some functional $\psi \in Y^*$, $\mathcal{R}(\partial^{0,1}F(\lambda^*,0)) = \{y \in Y | \psi(y) = 0\}$. (1.25) becomes, setting $\mu = \lambda - \lambda^*$,

$$\beta(\mu,t) := \psi(F(\lambda^* + \mu, tv^* + \gamma(\lambda^* + \mu, tv^*))) = 0. \tag{1.26}$$

Equation (1.26) is solved in a neighbourhood of $(0,0)$ by applying the implicit function theorem to the equation $h(\mu,t) = 0$, where h is a C^{k-1} (or real-analytic) function given by

$$h(\mu,t) = \begin{cases} \dfrac{\beta(\mu,t)}{t} & t \neq 0 \\ \beta^{(0,1)}(\mu,t) & t = 0 \end{cases}, \tag{1.27}$$

which is equivalent, for $t \neq 0$, to $\beta(\mu,t) = 0$. $h(\mu,t) = 0$ is solved by a unique C^{k-1} (or real-analytic) function μ on a neighbourhood $(-\varepsilon,\varepsilon)$ of 0 such that $\mu(0) = 0$; the set of nontrivial solutions of (1.24) is given, in a neighbourhood of $(\lambda^*,0)$ as $\{\mathcal{B}(t) | t \in (-\varepsilon,\varepsilon) \setminus \{0\}\}$, where $\mathcal{B}$ is a C^{k-1} (or real-analytic) function from $(-\varepsilon,\varepsilon)$ to U given by

$$\mathcal{B}(t) = (\lambda^* + \mu(t), tv^* + \gamma(\lambda^* + \mu(t), tv^*)) = (\Lambda(t), W(t)). \tag{1.28}$$

The behaviour of the function μ at 0 determines the behaviour of the curve of nontrivial solutions near zero: let m denote the lowest order derivative of μ which is nonzero at 0 (if such exists). The curve is said to be:

supercritical if m is even and $\mu^{(m)}(0) > 0$ (in this case, all solutions (λ, x) on $\mathfrak{B}$ in a neighbourhood of $(\lambda^*,0)$ are such that $\lambda \geq \lambda^*$);

subcritical if m is even and $\mu^{(m)}(0) < 0$ (in this case, all solutions (λ, x) on $\mathfrak{B}$ in a neighbourhood of $(\lambda^*,0)$ are such that $\lambda \leq \lambda^*$);

transcritical if m is odd (in this case there are solutions (λ_1, x_1) and (λ_2, x_2) in every neighbourhood of $(\lambda^*,0)$ on $\mathfrak{B}$ such that $\lambda_1 < \lambda^* < \lambda_2$).

The lowest order derivative of μ, nonzero at 0 is found by the following result, which is proved by a simple induction argument.

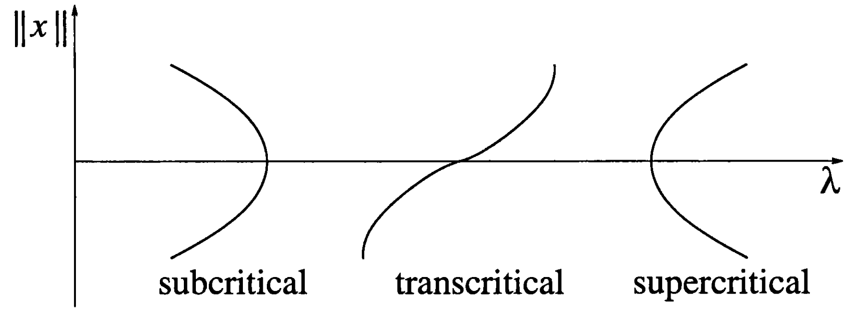

Figure 1-1: Diagram of types of bifurcations; the vertical axis is for illustration only

Theorem 1.24. *If $F \in C^{n+1}(U,Y)$ and $\mu(0) = \cdots = \mu^{(n-1)}(0) = 0$ then*

$$\mu^{(n)}(0) = -\frac{\beta^{(0,n+1)}(0,0)}{(n+1)\beta^{(1,1)}(0,0)}.$$

1.7 Global analytic bifurcation

In global analytic bifurcation theory, we attempt to continue analytic curves of solutions of (1.24) globally, in the case where $F : U \to Y$ is real-analytic. The abstract theorem which is the basis for Chapter 4 is below. We follow [6]:

Theorem 1.25. *(Global analytic bifurcation theorem) Suppose $F : U \to Y$ is real-analytic on U, and let $\mathcal{B} : (-\varepsilon, \varepsilon) \to U$ $(\varepsilon > 0)$ be a real-analytic curve of solutions of $F(\lambda, x) = 0$. Let $\mathfrak{B}^+ = \{\mathcal{B}(t) | t \in [0, \epsilon)\}$, and let $\mathcal{B}(t) = (\Lambda(t), W(t))$. Suppose also that*

I $\partial^{0,1}F(\lambda, x)$ is Fredholm with index 0 for all $(\lambda, x) \in U$ for which $F(\lambda, x) = 0$.

II ε is sufficiently small that

(a) $\mathrm{d}W(t)1 \neq 0$ for all $t \in (-\varepsilon, \varepsilon)$.

(b) $\Lambda' \not\equiv 0$ on $(-\varepsilon, \varepsilon)$.

III Closed bounded sets of solutions of $F(\lambda, x) = 0$ are compact in $\mathbb{R} \times X$.

Then there exists a continuous locally injective extension $\tilde{\mathfrak{B}}$ of $\mathfrak{B}^+$ such that *i* $\tilde{\mathfrak{B}} = \{\mathcal{B}(t) | t \in [0, \infty)\}$, *where* $\mathcal{B} : [0, \infty) \to U$.

ii $F(\mathcal{B}(t)) = 0$ *for all* $t \geq 0$.

iii *The set* $\{t \geq 0 | \mathcal{N}(\partial^{0,1} F(\mathcal{B}(t))) \neq \{0\}\}$ *has no accumulation points.*

iv *For each* $t^* > 0$ *there exists* $\delta^* \in (0, t^*)$ *and* $\sigma^* : (-\delta^*, \delta^*) \to \mathbb{R} \times X$ *such that* σ^* *is real-analytic and*

$$\{\mathcal{B}(t) || t - t^*| < \delta^*\} = \{\sigma^*(t) || t - t^*| < \delta^*\}.$$

v *One of the following occurs:*

a $||\mathcal{B}(t)||_{\mathbb{R} \times X} \to \infty$ *as* $t \to \infty$.

b $\mathcal{B}(t)$ *approaches* ∂U *as* $t \to \infty$.

c $\tilde{\mathfrak{B}}$ *is a closed loop: i.e. for some* $T > 0$, $\tilde{\mathfrak{B}} = \{\mathcal{B}(t) | t \in [0, T]\}$, *where* $\mathcal{B}(T) = \mathcal{B}(0) = (\lambda^*, 0)$.

Chapter 2

Regularity Theory

In this chapter we give local regularity results for solutions $w \in W^{1,1}_{2\pi}$ of (1.1). The main result is the following:

Theorem 2.1. *Let $f \in C^0(J)$ (where $J \subset \mathbb{R}$), and $c \in \mathbb{R}$. Let also $\hat{J}$ be an open subset of J, and let $w \in W^{1,1}_{2\pi}$ be a solution of (1.1) such that $\mathcal{C}w' \in L^1_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$.*

(I) If $f \in C^{n,\beta}(\hat{J} \setminus \mathcal{N}(f))$, where $\beta \in (0,1]$ and $n \in \mathbb{N} \cup \{0\}$, then

$$w \in C^{n+1,\alpha}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))) \text{ for all } \alpha \in (0,\beta) \tag{2.1}$$

if $n \in \{0,1\}$ or $\beta = 1$, and

$$w \in C^{n+1,\beta}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))) \tag{2.2}$$

if $n \in \mathbb{N} \setminus \{1\}$ and $\beta < 1$.

(II) If f is real-analytic on $\hat{J} \setminus \mathcal{N}(f)$ then w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(\mathcal{Z})$, where

$$\mathcal{Z} = f(w)\{w'^2 + (1 + \mathcal{C}w')^2\}. \tag{2.3}$$

[Note that as w is continuous (being in $W^{1,1}_{2\pi}$) and $\hat{J}$ is open, $w^{-1}(\hat{J})$ is open also; and since $f(w)$ is continuous, $\mathcal{N}(f(w))$ is closed. Hence $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ is open.]

This is proved in the first instance via Riemann-Hilbert theory (Section 2.1), which considers problems of the form $\Phi^* = a\overline{\Psi^*}$, where $a : S^1 \to \mathbb{C}$ is a given function and the holomorphic functions Φ and Ψ on $\mathcal{D}$ are to be found. It allows us to infer that $\mathcal{Z}$ is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$, and hence that $w' \in L^\infty_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$. In Section 2.2, an induction argument involving an operator $\mathcal{F}$ on $W^{1,1}_{2\pi}$, gives the first part of the result. In Section 2.3, further consideration of Riemann Hilbert theory, together with an application of Lewy's theorem (Theorem 1.13), gives the second part.

It should be noted that this result is extremely general: we shall show that it covers solutions $w \in W^{1,1}_{2\pi}$ of (1.1) under each of the following conditions:

(A) $w \in W^{1,p}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ for some $p > 1$.

(B) $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$.

(C) f is monotone on $\mathcal{R}(w)$.

In particular, it covers solutions $w \in W^{1,1}_{2\pi}$ of the Stokes wave equation (1.2). Using a separate argument, we also show in Section 2.3 that under the hypotheses of Theorem 2.1 (II), w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$. Together with Theorem 2.1 (II), this indicates that it is likely that under the same hypotheses, w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$; however we are at present unable to prove this.

In the theory of Stokes waves, the case where $\mathcal{Z} \equiv c$, the Bernoulli condition, is of great importance; for solutions of (1.2) which satisfy the Bernoulli condition describe a Stokes wave. In the present context, this condition (together with the condition $c \neq 0$ and the hypotheses of Theorem 2.1 (II)) gives us that w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$; and ensures that $f(w)$ has the same sign almost everywhere. The latter fact will be of importance in Chapter 4. In Section 2.4 we give, in case (B), sufficient conditions for the Bernoulli condition to hold and a necessary and sufficient condition for w to be [everywhere] real-analytic.

2.1 Riemann-Hilbert theory

In this section we show that (1.1) reduces to a Riemann-Hilbert problem of the form $U^* = f(w)\overline{W^*}$ where $U, W \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$, and show that each solution $w \in W^{1,1}_{2\pi}$ of (1.1) such that $\mathcal{C}w' \in L^1_{\text{loc}}(I)$ (where $I \subset S^1$ is open) is in $W^{1,\infty}_{\text{loc}}(I \setminus \mathcal{N}(f(w)))$. We begin with the following generalisation of Carleman's theorem ([12], p.64), to be found in [18].

Theorem 2.2. *Let $I \subset S^1$ be open, and let $\Phi, \Psi \in \mathcal{H}^p_{\mathbb{C}}$ (for some $p > 0$) satisfy $\Psi^* = \overline{\Phi^*}$ almost everywhere on I. If $|\Phi^*|, |\Psi^*| \in L^1_{\text{loc}}(I)$ then Φ has a bounded analytic extension $\hat{\Phi} : I^* \cup (\mathbb{C} \setminus \partial\mathcal{D}) \to \mathbb{C}$ given by $\hat{\Phi}(z) = \overline{\Psi(1/\overline{z})}$ ($|z| > 1$); and similarly for Ψ. In particular, $\Phi^*\Psi^*$ is real-analytic on I; and if $I = S^1$ then Φ and Ψ are constant functions.*

An immediate consequence of Theorem 2.2 is

Theorem 2.3. *Let $f \in C^0(J)$ and $c \in \mathbb{R}$. Let $w \in W^{1,1}_{2\pi}$ be a solution of (1.1).*

(1) If $c \neq 0$ then $f(w(t)) \neq 0$ almost everywhere. If $c = 0$ then either $f(w) \equiv 0$ or $f(w(t)) \neq 0$ almost everywhere.

(2) If $\mathcal{C}w' \in L^1_{\text{loc}}(I)$ for some open $I \subset S^1$ then $\mathcal{Z}$, given by (2.3), is real-analytic on $I \setminus \mathcal{N}(f(w))$.

Proof. Let $u = f(w)w' \in L^1_{2\pi}$. Then by property (iii) of Hardy spaces, we have that for some $U, W \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$ with $W(0) = i$, $W^* = w' + i(1 + \mathcal{C}w')$ and $U^* = u + i(-c + \mathcal{C}u)$. We have

$$U(0) = -ic + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mathrm{d}}{\mathrm{d}t}\phi(w(t))\ \mathrm{d}t = -ic,$$

where ϕ is a primitive of f. (1.1) becomes the Riemann-Hilbert problem

$$U^* = f(w)w' - if(w)(1 + \mathcal{C}w') = f(w)\overline{W^*}. \tag{2.4}$$

(1) Suppose $c \neq 0$. Then $U \not\equiv 0 \not\equiv W$, so $|U^*|, |W^*| \in \text{Log}$ by property (i) of $\mathcal{H}^p_{\mathbb{C}}$. Hence $\log|f(w)| = \log|U^*| - \log|W^*| \in L^1_{2\pi}$; so $f(w(t)) \neq 0$ for almost every t.

Suppose $c = 0$ and $f(w) \not\equiv 0$. Since $W \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}} \setminus \{0\}$, $|W^*| \in \text{Log}$; so $W^*(t) \neq 0$ almost everywhere. Now $f(w(t)) \neq 0$ on a set of positive measure (since $f(w)$ is a continuous, nonzero function); so $U^* \not\equiv 0$ by (2.4). Hence, as with W^*, $U^*(t) \neq 0$ almost everywhere. Hence as above, $f(w) \in \text{Log}$ and $f(w(t)) \neq 0$ almost everywhere.

(2) If $f(w) \equiv 0$ then the result is trivial. Otherwise by (1), $f(w) \in \text{Log}$ Suppose $\mathcal{C}w' \in L^1_{\text{loc}}(I)$. Let $\sigma = \text{sign}(f(w))$ and $H = \mathcal{O}\sqrt{|f(w)|}$. Then by property (b) of outer functions, $|H^*|^2 = |f(w)|$, and

$$\left(\frac{U}{H}\right)^* = \sigma\overline{(HW)^*}.$$

$f(w) \in L^\infty_{2\pi}$, so by property (c) of outer functions, $H \in \mathcal{H}^\infty_{\mathbb{C}}$ Since $W \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$, we have by applying Hölder's inequality to the functions $H_r W_r$, and by the definition of the spaces $\mathcal{H}^p_{\mathbb{C}}$ (Definition 1.11) that $HW \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$

Since $|H^*| \in L^\infty_{2\pi}$ and $|W^*| \in \cap_{p<1}L^p_{2\pi} \cap L^1_{\text{loc}}(I)$, we have by Hölder's inequality that $H^*W^* \in \cap_{p<1}L^p_{2\pi} \cap L^1_{\text{loc}}(I)$.

By properties (d), (c) and (b) of outer functions,

$$\left|\frac{U(z)}{H(z)}\right| \leq \left|\frac{\mathcal{O}(U^*)(z)}{H(z)}\right| = \left|\mathcal{O}\left(\frac{U^*}{H^*}\right)(z)\right| = |\mathcal{O}(H^*W^*)(z)|, \tag{2.5}$$

so $U/H \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$ also with $|(U/H)^*| \in L^1_{\text{loc}}(I)$.

Now $f(w)$ is continuous, since $f \in C^0(J)$ and $w \in W^{1,1}_{2\pi}$; so the sets $S^+ := \{t \in I | f(w(t)) > 0\}$ and $S^- := \{t \in I | f(w(t)) < 0\}$ are open. On S^+, we have $(U/H)^* = \overline{(HW)^*}$, so by Theorem 2.2, U/H and HW have analytic extensions to $(S^+)^* \cup (\mathbb{C} \setminus \partial\mathcal{D})$, and

$$\left(\frac{U}{H}\right)^* (HW)^* = U^*W^* = f(w)|W^*|^2 = \mathcal{Z}$$

is real-analytic on S^+. Similarly $\mathcal{Z}$ is real-analytic on S^-. The result follows. $\square$

Corollary 2.4. *Suppose f, c and w are as in Theorem 2.3. Then $w \in W^{1,\infty}_{\mathrm{loc}}(I \setminus \mathcal{N}(f(w)))$.*

Proof. By Theorem 2.3, $\mathcal{Z} = f(w)\{w'^2 + (1 + \mathcal{C}w'^2)\}$ is real-analytic on $I \setminus \mathcal{N}(f(w))$. Hence $\mathcal{Z}/f(w) \in L^\infty_{\mathrm{loc}}(I \setminus \mathcal{N}(f(w)))$. Since w'^2 and $(1 + \mathcal{C}w')^2$ are nonnegative, it follows that w'^2, and hence w', is in $L^\infty_{\mathrm{loc}}(I \setminus \mathcal{N}(f(w)))$ also. $\square$

2.2 The operator $\mathcal{F}$: Hölder continuity

In this section we give an inductive proof of the first part of Theorem 2.1, with base cases $n = 0$ (Theorem 2.6) and $n = 1$ (Theorem 2.9). The inductive step follows at the end of the section. We define first the operator $\mathcal{F}$.

Let $f \in C^0(J)$ where $J \subset \mathbb{R}$. We define, for $u \in W^{1,1}_{2\pi}$ with $\mathcal{R}(u) \subset J$,

$$\mathcal{F}(u)(x) = f(u(x))\mathcal{C}u'(x) - \mathcal{C}(u'f(u))(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(u(x)) - f(u(y))}{\tan\frac{x-y}{2}} u'(y)\,dy. \tag{2.6}$$

Note that $\mathcal{F}(u)$ is defined only as a Cauchy principal value integral, being the difference of two such integrals. Note also that $\mathcal{F}$ commutes with translations, in the following sense: for $\tau \in \mathbb{R}$, let $u_\tau \in W^{1,1}_{2\pi}$ be a translation of u:

$$u_\tau(x) = u(x + \tau).$$

Then $\mathcal{F}(u_\tau)(x) = \mathcal{F}(u)(x + \tau)$. This follows from (2.6) since $\mathcal{C}$ commutes with translations also. In particular, $\mathcal{F}(u)$ is 2π-periodic.

In the case where $f \in C^0(J) \cap C^{0,\beta}(\hat{J})$ (see Theorem 2.1 for notation) and u satisfies condition (A) above, it can be shown that $\mathcal{F}(u) \in L^q_{\mathrm{loc}}(u^{-1}(\hat{J}))$ for some $q > p$. Further, using a bootstrapping argument involving $\mathcal{F}$, it can be shown, *without using Riemann-Hilbert Theory*, that if $f \in C^0(J) \cap C^{0,\beta}(\hat{J} \backslash \mathcal{N}(f))$ and w satisfies condition (A) and is a solution of (1.1) then $w \in C^{0,\alpha}_{\mathrm{loc}}(w^{-1}(\hat{J}) \backslash \mathcal{N}(f(w)))$ for all $\alpha \in (0,1)$. This argument is presented in Appendix B.

We present first an alternative expression for $\mathcal{F}(u)$. We have

$$
\begin{aligned}
\mathcal{F}(u)(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\{f(u(x)) - f(u(y))\}\, u'(y)}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\frac{\mathrm{d}}{\mathrm{d}y}\{f(u(x))u(y) - \phi(u(y))\}}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\frac{\mathrm{d}}{\mathrm{d}y}\{f(u(x))\,(u(y) - u(x)) - (\phi(u(y)) - \phi(u(x)))\}}{\tan\frac{x-y}{2}}\,\mathrm{d}y,
\end{aligned}
\tag{2.7}
$$

where ϕ is a primitive of f. Now $u \in W^{1,1}_{2\pi}$ so for almost every $x \in S^1$, u has a classical derivative at x. For such x we have by the chain rule (1.7) that as $y \to x$,

$$
\left| \frac{\phi(u(x)) - \phi(u(y)) + [u(y) - u(x)]f(u(x))}{x-y} \right| \to 0;
$$

so we may integrate (2.7) by parts: for almost every x,

$$
\begin{aligned}
&\mathcal{F}(u)(x) \\
&= \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left(\int_{x-\pi}^{x-\varepsilon} + \int_{x+\varepsilon}^{x+\pi}\right) \frac{\frac{\mathrm{d}}{\mathrm{d}y}\{f(u(x))\,(u(y)-u(x)) - \phi(u(y)) + \phi(u(x))\}}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left\{ \left[\frac{f(u(x))\,(u(y) - u(x)) - (\phi(u(y)) - \phi(u(x)))}{\tan\frac{x-y}{2}}\right]_{x-\pi}^{x-\varepsilon} \right. \\
&\quad + \left[\frac{f(u(x))\,(u(y) - u(x)) - (\phi(u(y)) - \phi(u(x)))}{\tan\frac{x-y}{2}}\right]_{x+\varepsilon}^{x+\pi} \\
&\quad \left. -\frac{1}{2}\left(\int_{x-\pi}^{x-\varepsilon} + \int_{x+\varepsilon}^{x+\pi}\right) \frac{f(u(x))\,(u(y) - u(x)) - (\phi(u(y)) - \phi(u(x)))}{\sin^2\frac{x-y}{2}}\,\mathrm{d}y \right\} \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\phi(u(x)) - \phi(u(y)) + (u(y) - u(x))f(u(x))}{\sin^2\frac{x-y}{2}}\,\mathrm{d}y \\
&= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{G(u(x), u(x-y))}{\sin^2\frac{y}{2}}\,\mathrm{d}y,
\end{aligned}
\tag{2.8}
$$

where $G : J^2 \to \mathbb{R}$ is given by

$$G(s,t) = \phi(s) - \phi(t) + (t-s)f(s).$$

We introduce next some notation. Suppose $E_1 \subset S^1$ and E_2 is a compact segment of E_1°. We define the following

$$\delta(E_1, E_2) = \begin{cases} \pi & E_1 = S^1 \\ \frac{1}{2}\text{dist}(E_2, \partial E_1) & E_1 \neq S^1, \end{cases} \tag{2.9}$$

and

$$\mathcal{E}(E_1, E_2) = \{\varphi + \psi \in S^1 | \varphi \in E_2, |\psi| \leq \delta\} \tag{2.10}$$

[note that if $E_1 = S^1$ then $\mathcal{E}(E_1, E_2) = S^1$ also]. We define finally a smooth function $\xi_{E_1,E_2} : S^1 \to [0,1]$ such that $\xi_{E_1,E_2} \equiv 1$ on $\mathcal{E}(E_1, E_2)$ and $\xi_{E_1,E_2} \equiv 0$ on $S^1 \setminus E_1$.

We now begin the proof of Theorem 2.1 (I), and start with the case $n = 0$.

Lemma 2.5. *(generalisation of lemma 3.5 in [4]) Suppose $f \in C^0(J) \cap C^{0,\beta}(\hat{J})$, where J, $\hat{J}$ and β are as in Theorem 2.1. If $u \in W^{1,1}_{2\pi} \cap C^{0,\alpha}(E)$, where $\alpha \in \left(\frac{1}{1+\beta}, 1\right)$, $E \subset u^{-1}(\hat{J})$ and $\mathcal{R}(u) \subset J$, then $\mathcal{F}(u) \in C^{0,\alpha(1+\beta)-1}_{\text{loc}}(E^\circ)$.*

Proof. Choose a compact segment $\tilde{E}$ of E°. Since $\tilde{E}$ is compact, it will suffice to show that for each $x \in \tilde{E}$, $\mathcal{F}(u)$ is $C^{0,\alpha(1+\beta)-1}$ in a neighbourhood of x. For almost every $t \in S^1$ we have that,

$$\mathcal{F}(u)(t) = -\frac{1}{4\pi}\left\{\int_{-\delta}^{\delta} + \int_X\right\} \frac{G(u(t), u(t-y))}{\sin^2 \frac{y}{2}}\, \mathrm{d}y = -\frac{1}{4\pi}\{D(t) + S(t)\}, \tag{2.11}$$

say, where $\delta = \delta(E, \tilde{E})$ (as given in (2.9)) and $X = S^1 \setminus [-\delta, \delta]$. Note that if $E = S^1$, then $X = \emptyset$ and $S \equiv 0$. We shall consider the regularity of D and S separately.

Let $|h| \leq \frac{\delta}{2}$. Then

$$\begin{aligned}
|D(x+h) - D(x)| &= \left| \int_{-\delta}^{\delta} \frac{G(u(x+h), u(x+h-y)) - G(u(x), u(x-y))}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \right| \\
&\leq \int_{-|h|}^{|h|} \frac{|G(u(x+h), u(x+h-y))|}{\sin^2 \frac{y}{2}} + \frac{|G(u(x), u(x-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \\
&\quad + \int_{Y} \frac{|G(u(x+h), u(x+h-y)) - G(u(x), u(x-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y,
\end{aligned} \tag{2.12}$$

where $Y = [-\delta, \delta] \setminus [-|h|, |h|]$. We consider each of these integrals in turn.

By the mean value theorem, for all $x \in \tilde{E}$ and $|y| \leq |h|$, there exists $\xi \in [u(x-y), u(x)]$ such that $\phi(u(x)) - \phi(u(x-y)) = f(\xi)\{u(x) - u(x-y)\}$. Hence for some constant C independent of x and y,

$$|G(u(x), u(x-y))| = |f(\xi) - f(u(x))||u(x) - u(x-y)| \leq C|y|^{\alpha(1+\beta)},$$

(by Theorem 1.4 (iii), since $f \in C^{0,\beta}(\hat{J})$ and u is $C^{0,\alpha}$ on $[x, x-y] \subset [x-|h|, x+|h|] \subset [x-\delta, x+\delta] \subset E$). Similarly, if $|y| \leq |h|$, then $|G(u(x+h), u(x+h-y))| \leq C|y|^{\alpha(1+\beta)}$ for some constant C, independent of x, y and h (we denote all such constants C in this proof), since u is $C^{0,\alpha}$ on $[x+h, x+h-y] \subset [x-2|h|, x+2|h|] \subset [x-\delta, x+\delta] \subset E$. It follows that

$$\begin{aligned}
&\int_{-|h|}^{|h|} \frac{|G(u(x+h), u(x+h-y))|}{\sin^2 \frac{y}{2}} + \frac{|G(u(x), u(x-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \\
&\leq C \int_{-|h|}^{|h|} |y|^{\alpha(1+\beta)-2} \, \mathrm{d}y \leq C|h|^{\alpha(1+\beta)-1}.
\end{aligned} \tag{2.13}$$

In order to bound the second integral in (2.12), we rearrange $K(x, h, y) :=$

$G(u(x+h), u(x+h-y)) - G(u(x), u(x-y))$:

$$\begin{aligned}
K(x,h,y) &= \{\phi(u(x+h)) - \phi(u(x))\} - \{\phi(u(x+h-y)) - \phi(u(x-y))\} \\
&\quad + \{[u(x+h-y) - u(x+h)] - [u(x-y) - u(x)]\} f(u(x+h)) \\
&\quad + \{u(x-y) - u(x)\}\{f(u(x+h)) - f(u(x))\}. \qquad (2.14)
\end{aligned}$$

By the mean value theorem (applied to ϕ) and the intermediate value theorem (applied to u),

$$\begin{aligned}
\phi(u(x+h)) - \phi(u(x)) &= f(u(t_0))\{u(x+h) - u(x)\} \\
\phi(u(x+h-y)) - \phi(u(x-y)) &= f(u(t_1))\{u(x+h-y) - u(x-y)\}
\end{aligned}$$

for some $t_0 \in [x, x+h]$ and $t_1 \in [x-y, x+h-y]$. It follows that

$$\begin{aligned}
K(x,h,y) &= \{[u(x+h) - u(x)] - [u(x+h-y) - u(x-y)]\} f(u(t_0)) \\
&\quad + \{u(x+h-y) - u(x-y)\}\{f(u(t_0)) - f(u(t_1))\} \\
&\quad + \{[u(x+h-y) - u(x+h)] - [u(x-y) - u(x)]\} f(u(x+h)) \\
&\quad + \{u(x-y) - u(x)\}\{f(u(x+h)) - f(u(x))\}
\end{aligned}$$

$$\begin{aligned}
&= \{[u(x+h) - u(x+h-y)] - [u(x) - u(x-y)]\}\{f(u(t_0)) - f(u(x+h))\} \\
&\quad + \{u(x+h-y) - u(x-y)\}\{f(u(t_0)) - f(u(t_1))\} \\
&\quad + \{u(x-y) - u(x)\}\{f(u(x+h)) - f(u(x))\}.
\end{aligned}$$

Suppose $y \in Y$. Then by Theorem 1.4 (iii),

$$\begin{aligned}
&|[u(x+h) - u(x+h-y)] - [u(x) - u(x-y)]||f(u(t_0)) - f(u(x+h))| \\
&\leq C|y|^\alpha |t_0 - x - h|^{\alpha\beta} \leq C|y|^\alpha |h|^{\alpha\beta},
\end{aligned}$$

since $f \in C^{0,\beta}(\hat{J})$, u is $C^{0,\alpha}$ on $[x-\delta-|h|, x+\delta+|h|] \subset E$ and $|t_0 - x - h| \leq |h|$. Similarly,

$$|u(x+h-y)-u(x-y)||f(u(t_0))-f(u(t_1))| \leq C|h|^{\alpha}|t_0-t_1|^{\alpha\beta} \leq C|y|^{\alpha\beta}|h|^{\alpha},$$

since for $y \in Y$, $|t_0 - t_1| \leq |h| + |y| \leq 2|y|$; and

$$|u(x-y)-u(x)||f(u(x+h))-f(u(x))| \leq C|y|^{\alpha}|h|^{\alpha\beta}. \tag{2.15}$$

It follows that $|K(x,h,y)| \leq C|y|^{\alpha\beta}|h|^{\alpha} + C|y|^{\alpha}|h|^{\alpha\beta}$, so

$$\begin{aligned} \int_Y \frac{|K(x,h,y)|}{\sin^2 \frac{y}{2}}\, dy &\leq C|h|^{\alpha} \int_Y \frac{|y|^{\alpha\beta}}{\sin^2 \frac{y}{2}}\, dy + C|h|^{\alpha\beta} \int_Y \frac{|y|^{\alpha}}{\sin^2 \frac{y}{2}}\, dy \\ &\leq C|h|^{\alpha} \int_Y |y|^{\alpha\beta-2}\, dy + C|h|^{\alpha\beta} \int_Y |y|^{\alpha-2}\, dy, \end{aligned}$$

since for $y \in Y$, $|\sin \frac{y}{2}| \geq \frac{1}{2\pi}|y|$. Hence

$$\begin{aligned} \int_Y \frac{|K(x,h,y)|}{\sin^2 \frac{y}{2}}\, dy &\leq C|h|^{\alpha} \left[y^{\alpha\beta-1}\right]_{\delta}^{|h|} + C|h|^{\alpha\beta} \left[y^{\alpha-1}\right]_{\delta}^{|h|} \\ &= C|h|^{\alpha} \left\{|h|^{\alpha\beta-1} - \delta^{\alpha\beta-1}\right\} + C|h|^{\alpha\beta} \left\{|h|^{\alpha-1} - \delta^{\alpha-1}\right\} \\ &\leq C|h|^{\alpha(1+\beta)-1}, \end{aligned}$$

as required. This establishes the regularity of D.

Now for $t \in S^1$,

$$\begin{aligned} S(t) &= \{\phi(u(t)) - u(t) f(u(t))\} \int_X \frac{dy}{\sin^2 \frac{y}{2}} \\ &\quad + f(u(t)) \int_X \frac{u(t-y)}{\sin^2 \frac{y}{2}}\, dy - \int_X \frac{\phi(u(t-y))}{\sin^2 \frac{y}{2}}\, dy \\ &= S_1(t) + S_2(t) + S_3(t), \end{aligned} \tag{2.16}$$

say. Since $y \mapsto \sin^2 \frac{y}{2}$ is bounded away from 0 on X, $y \mapsto \operatorname{cosec}^2 \frac{y}{2}$ is smooth on

X. Since $f \in C^{0,\beta}(\hat{J})$ and $u \in C^{0,\alpha}(E)$, it follows from Theorem 1.4 (i) and (iii) that $S_1 \in C^{0,\alpha\beta}(E)$.

By Theorem 1.4 (iii), the function multiplying the integral in S_2 is $C^{0,\alpha\beta}(E)$. We show that the integral is $C^1_{2\pi}$:

$$\int_X \frac{u(t-y)}{\sin^2 \frac{y}{2}}\, \mathrm{d}y = \left[-U(t-y)\mathrm{cosec}^2\frac{y}{2}\right]_\delta^{2\pi-\delta} + \int_X U(t-y)\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{cosec}^2\frac{y}{2}\, \mathrm{d}y, \quad (2.17)$$

where $U(t) = \int_0^t u(s)\, \mathrm{d}s$ (note that $X = (\delta, 2\pi - \delta)$). Since $u \in W^{1,1}_{2\pi} \subset C^0_{2\pi}$, $U \in C^1_{2\pi}$, so as $y \mapsto \mathrm{cosec}^2\frac{y}{2}$ is smooth on X, both terms on the right hand side of (2.17) are $C^1_{2\pi}$, as required. Hence by Theorem 1.4 (i), $S_2 \in C^{0,\alpha\beta}(E)$. Similarly, $S_3 \in C^1_{2\pi}$.

Since for all $\alpha, \beta \leq 1$, $\alpha\beta \geq \alpha(1+\beta) - 1$, we have that $S \in C^{0,\alpha(1+\beta)-1}(E)$, as required. $\square$

Theorem 2.6. *Suppose $f \in C^0(J) \cap C^{0,\beta}(\hat{J} \setminus \mathcal{N}(f))$, where J, $\hat{J}$ and β are as in Theorem 2.1, and $c \in \mathbb{R}$. If w is as in Theorem 2.1 then $w \in C^{1,\alpha}_{\mathrm{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ for each $\alpha \in (0, \beta)$.*

Proof. Let E_4 be a compact segment of $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ and let, for $i \in \{1,2,3\}$, E_i be a compact segment of $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ such that $E_{i+1} \subset E_i^\circ$. We show that $w \in C^{1,\alpha}(E_4)$ for each $\alpha \in (0, \beta)$.

Since w satisfies (1.1),

$$\begin{aligned} f(w) - \mathcal{F}(w) &= f(w) - f(w)\mathcal{C}w' + \mathcal{C}(f(w)w') \\ &= f(w)(1 + \mathcal{C}w') + \mathcal{C}(f(w)w') - 2f(w)\mathcal{C}w' \\ &= c - 2f(w)\mathcal{C}w'. \end{aligned} \quad (2.18)$$

Since $f(w)$ is continuous (since $w \in W^{1,1}_{2\pi} \subset C^0_{2\pi}$), it is bounded away from 0 on

E_1. Hence by (2.18) we have that for almost every $x \in E_1$,

$$\mathcal{C}w'(x) = \frac{c + \mathcal{F}(w)(x)}{2f(w)(x)} - \frac{1}{2}. \tag{2.19}$$

By Corollary 2.4 and Hölder's inequality, $w \in W^{1,\infty}(E_1) \subset C^{0,1}(E_1)$. Since $E_1 \subset w^{-1}(\hat{J} \setminus \mathcal{N}(f(w)))$, we have by Lemma 2.5 that $\mathcal{F}(w) \in C^{0,\alpha(1+\beta)-1}(E_2)$ for each $\alpha \in (0,1)$: i.e. $\mathcal{F}(w) \in C^{0,\alpha}(E_2)$ for each $\alpha \in (0,\beta)$.

Theorem 1.4 (iii) now yields $f(w) \in C^{0,\beta}(E_2)$. Since $f(w)$ is bounded away from 0 on E_2, we may apply Theorem 1.4 (ii) to (2.19), to conclude that $\mathcal{C}w' \in C^{0,\alpha}(E_2)$ for each $\alpha \in (0,\beta)$. Now

$$\mathcal{C}w' = \mathcal{C}(\xi w') + \mathcal{C}((1-\xi)w'), \tag{2.20}$$

where $\xi = \xi_{E_2,E_3}$ is as given below (2.10). By the proof of Theorem 1.8, $\mathcal{C}((1-\xi)w')$ is smooth on E_3, so $\mathcal{C}(\xi w') \in C^{0,\alpha}(E_3)$ for each $\alpha \in (0,\beta)$. Now $w' \in L^\infty(E_1)$, so $\xi w' \in L^\infty_{2\pi}$. It follows by the theorem of M. Riesz (Theorem 1.7) that $\mathcal{C}(\xi w') \in L^1_{2\pi}$, and hence by Theorem 1.8 (2) that $\xi w' = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(t)w'(t)\,\mathrm{d}t - \mathcal{C}^2(\xi w') \in C^{0,\alpha}(E_4)$ for each $\alpha \in (0,\beta)$. The result follows since on E_4, $w' \equiv \xi w'$. $\square$

That Lemma 2.5 holds only when $\alpha < 1$ is somewhat unsatisfactory, since it weakens the conclusions of Theorem 2.6 (and Theorem 2.1). However, as the following example shows, Lemma 2.5 is best possible.

Example 2.7. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|^\beta$, where $\beta \in (0,1]$; and let $u : S^1 \to \mathbb{R}$ be given by $u(x) = |x|$. Then $f \in C^{0,\beta}(\mathbb{R})$ and $u \in C^{0,1}_{2\pi}$. Let $0 \le h \le \frac{\pi}{2}$. By (2.11-2.13),

$$\mathcal{F}(u)(h) - \mathcal{F}(u)(0) = I_0 - \frac{1}{4\pi}\int_Y \frac{K(0,h,y)}{\sin^2\frac{y}{2}}\,\mathrm{d}y,$$

where $Y = S^1 \setminus [-h,h]$, $K(0,h,y)$ is given by (2.14) and $|I_0| \le Ch^\beta$ for some constant C, independent of h (we denote all such constants C in this example). Now

$$\int_Y \frac{K(0,h,y)}{\sin^2\frac{y}{2}}\,dy = \left\{\left(\int_{-\pi}^{h-\pi}+\int_{\pi-h}^{\pi}\right)+\left(\int_{h-\pi}^{-h}+\int_{h}^{\pi-h}\right)\right\}\frac{K(0,h,y)}{\sin^2\frac{y}{2}}\,dy = I_1 + I_2.$$

We examine first the size of I_1. By (2.14-2.15) we have that for $y \in Y$,

$$|K(0,h,y)| \leq C|y|^\beta h + C|y|h^\beta$$

for some constant C, independent of h and y [note that (2.14-2.15) hold for all $\alpha, \beta \in [0,1]$]; so we have that

$$|I_1| \leq C\left(\int_{-\pi}^{h-\pi}+\int_{\pi-h}^{\pi}\right)|y|^{\beta-2}h + |y|^{-1}h^\beta\,dy \leq Ch^\beta \log\frac{\pi}{\pi-h} \leq Ch^\beta.$$

Next we evaluate I_2. We have, for $x>0$, $\phi(x) = \int_0^x t^\beta\,dt = \frac{1}{\beta+1}x^{\beta+1}$, so

$$K(0,h,y) = \frac{1}{\beta+1}\{|y|^{\beta+1} - |h-y|^{\beta+1}\} + |h-y|h^\beta - \frac{\beta}{\beta+1}h^{\beta+1},$$

so that

$$\begin{aligned}
I_2 &= \left(\int_{h-\pi}^{-h}+\int_{h}^{\pi-h}\right)\frac{\frac{1}{\beta+1}\{|y|^{\beta+1}-|h-y|^{\beta+1}\}+|h-y|h^\beta-\frac{\beta}{\beta+1}h^{\beta+1}}{\sin^2\frac{y}{2}}\,dy\\
&= \kappa(h)\int_h^{\pi-h}\frac{\frac{1}{\beta+1}\{2y^{\beta+1}-(y-h)^{\beta+1}-(y+h)^{\beta+1}\}+2yh^\beta-\frac{2\beta}{\beta+1}h^{\beta+1}}{y^2}\,dy
\end{aligned}$$

for some $\kappa(h) \in [4,\pi^2]$, since for $y \in S^1$, $\frac{|y|}{\pi} \leq |\sin\frac{y}{2}| \leq \frac{|y|}{2}$. Let $t = \frac{y}{h}$. Then

$$\begin{aligned}
I_2 &= \kappa(h)h^\beta\int_1^{\frac{\pi}{h}-1}\frac{1}{t^2}\left(\frac{1}{\beta+1}\{2t^{\beta+1}-(t+1)^{\beta+1}-(t-1)^{\beta+1}\}+2t-\frac{2\beta}{\beta+1}\right)\,dt\\
&= \kappa(h)h^\beta\int_1^{\frac{\pi}{h}-1}\frac{2}{t}-\frac{t^{\beta-1}}{\beta+1}(\phi_1(1/t)+\phi_1(-1/t))-\frac{2\beta}{(\beta+1)t^2}\,dt, \qquad (2.21)
\end{aligned}$$

where $\phi_1(s)$ is the 0th order Taylor series remainder of the function $s \mapsto (1+s)^{\beta+1}$

about 0. We have by (1.19) that $|\phi_1(s)| = O(|s|)$ as $s \to 0$. It follows that each term on the right-hand side of (2.21) except

$$\kappa(h)h^\beta \int_1^{\frac{\pi}{h}-1} \frac{2}{t}\,\mathrm{d}t$$

is dominated by $C\kappa(h)h^\beta$. The latter term equals

$$2\kappa(h)h^\beta \log\left(\frac{\pi}{h} - 1\right).$$

Hence if $0 < h < \frac{\pi}{2}$ then $\mathcal{F}(u)(h) - \mathcal{F}(u)(0) = R_h + \kappa(h)h^\beta \log\left(\frac{\pi}{h} - 1\right)$, where $|R_h| \leq C\kappa(h)h^\beta$ and $\kappa(h)$ is bounded and bounded away from 0 as $h \to 0$. Since, as $h \to 0$, $\log\left(\frac{\pi}{h} - 1\right) \to \infty$, it follows that $\mathcal{F}(u) \notin C^{0,\beta}_{2\pi}$. ■

This concludes the case $n = 0$. In the case $n = 1$ we have

Lemma 2.8. *(generalisation of lemma 3.6 in [4]) Suppose $f \in C^0(J) \cap C^{1,\beta}(\hat{J})$, where J, $\hat{J}$ and β are as in Theorem 2.1. If $u \in W^{1,1}_{2\pi} \cap C^{1,\alpha}(E)$, with $\alpha \in (0,1]$, $E \subset u^{-1}(\hat{J})$ and $\mathcal{R}(u) \subset J$, then $\mathcal{F}(u) \in C^{1,\varepsilon}_{\mathrm{loc}}(E^\circ)$ for each $\varepsilon \in (0, \min\{\alpha, \beta\})$.*

Proof. Choose a compact segment, $\tilde{E}$ of E° and let δ, D and S be as in the proof of Lemma 2.5. Since $\tilde{E}$ is compact, it will suffice to show that for each $x \in \tilde{E}$, D and S are sufficiently regular in a neighbourhood of x; since $\mathcal{F}$ commutes with translations, it will suffice to consider the case $x = 0 \in \tilde{E}$.

We show first that D is differentiable in a neighbourhood of 0, and that

$$D'(k) = \int_{-\delta}^{\delta} \frac{\frac{\partial}{\partial k} G(u(k), u(k-y))}{\sin^2 \frac{y}{2}}\,\mathrm{d}y \tag{2.22}$$

for all k in a neighbourhood of 0 (we show that the integrand on the right-hand side of (2.22) is in $L^1(-\delta, \delta)$ in the course of the proof). It will suffice to show that for all h and k in a neighbourhood of 0,

$$\begin{aligned}
&\left| D(k+h) - D(k) - h \int_{-\delta}^{\delta} \frac{\frac{\partial}{\partial k} G(u(k), u(k-y))}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \right| \\
&= \left| \int_{-\delta}^{\delta} \frac{G(u(k+h), u(k+h-y)) - G(u(k), u(k-y)) - h\frac{\partial}{\partial k} G(u(k), u(k-y))}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \right| \\
&\leq C|h|^{1+\min\{\alpha,\beta\}} |\log |Ch||, \qquad (2.23)
\end{aligned}$$

where C is a constant independent of k, y and h (we denote all such constants C in this proof). Note that (together with the uniform boundedness of D' in a neighbourhood of 0), this will in fact prove that for all $\varepsilon \in (0, \min\{\alpha, \beta\})$, D is $C^{1,\varepsilon}$ in a neighbourhood of 0, since for k_1 and k_2 in a neighbourhood of 0 we shall then have

$$\begin{aligned}
\frac{|D'(k_1) - D'(k_2)|}{|k_1 - k_2|^{\varepsilon}} &\leq \frac{1}{|k_1 - k_2|^{\varepsilon}} \left| D'(k_1) - \frac{D(k_1) - D(k_2)}{k_1 - k_2} \right| \\
&\quad + \frac{1}{|k_1 - k_2|^{\varepsilon}} \left| \frac{D(k_2) - D(k_1)}{k_2 - k_1} - D'(k_2) \right| \\
&\leq C|k_1 - k_2|^{\min\{\alpha,\beta\}-\varepsilon} |\log |C'(k_1 - k_2)|| \to 0 \quad (2.24)
\end{aligned}$$

for some constant C, independent of k_1 and k_2, as $k_1 - k_2 \to 0$.

Let $|h|, |k| \leq \min\{1, \frac{\delta}{3}\}$ and $Y = [-\delta, \delta] \setminus [-|h|, |h|]$. Then

$$\begin{aligned}
&\left| \int_{-\delta}^{\delta} \frac{G(u(k+h), u(k+h-y)) - G(u(k), u(k-y)) - h\frac{\partial}{\partial k} G(u(k), u(k-y))}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \right| \\
&\leq \int_{Y} \frac{|G(u(k+h), u(k+h-y)) - G(u(k), u(k-y)) - h\frac{\partial}{\partial k} G(u(k), u(k-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \\
&\quad + \int_{-|h|}^{|h|} \frac{|G(u(k+h), u(k+h-y)) - G(u(k), u(k-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y \\
&\quad + |h| \int_{-|h|}^{|h|} \frac{|\frac{\partial}{\partial k} G(u(k), u(k-y))|}{\sin^2 \frac{y}{2}} \, \mathrm{d}y. \qquad (2.25)
\end{aligned}$$

Consider the third term on the right-hand side of (2.25).

If $|y| \le \delta$, then $[k, k-y] \subset \left[-\frac{4}{3}\delta, \frac{4}{3}\delta\right] \subset E$, so by the mean value theorem, we have that for some $\zeta \in [u(k), u(k-y)]$, $f(u(k)) - f(u(k-y)) = f'(\zeta)\{u(k) - u(k-y)\}$. Hence if $|y| \le \delta$, then

$$\begin{aligned}
&\frac{\partial}{\partial k} G(u(k), u(k-y)) \\
&\quad = \{f(u(k)) - f(u(k-y))\}u'(k-y) + \{u(k-y) - u(k)\}f'(u(k))u'(k) \\
&\quad = \{f'(\zeta)u'(k-y) - f'(u(k))u'(k)\}\{u(k) - u(k-y)\} \\
&\quad = \{f'(\zeta)[u'(k-y) - u'(k)] + [f'(\zeta) - f'(u(k))]u'(k)\}\{u(k) - u(k-y)\},
\end{aligned} \tag{2.26}$$

so by Theorem 1.4 (iii),

$$\begin{aligned}
\left|\frac{\partial}{\partial k} G(u(k), u(k-y))\right| &\le |f'(\zeta)||u'(k-y) - u'(k)||u(k) - u(k-y)| \\
&\quad + |f'(\zeta) - f'(u(k))||u'(k)||u(k) - u(k-y)| \\
&\le C|y|\{|y|^\alpha + |y|^\beta\} \le C|y|^{1+\min\{\alpha,\beta\}}
\end{aligned}$$

for all y with $|y| \le \delta$. Hence

$$|h| \int_{-|h|}^{|h|} \frac{|\frac{\partial}{\partial k} G(u(k), u(k-y))|}{\sin^2 \frac{y}{2}}\, dy \le C|h| \int_{-|h|}^{|h|} |y|^{\min\{\alpha,\beta\}-1}\, dy = C|h|^{1+\min\{\alpha,\beta\}}, \tag{2.27}$$

since for $y \in [-|h|, |h|]$, $|\sin \frac{y}{2}| \ge \frac{1}{2\pi}|y|$. A similar calculation shows that the integrand on the right-hand side of (2.22) is in $L^1(-\delta, \delta)$.

For fixed $y \in [-\delta, \delta]$, $k \mapsto G(u(k), u(k-y))$ is clearly continuously differentiable on $[-\delta, \delta]$. Let $\Phi_y(k) = G(u(k), u(k-y))$. Then by the mean value theorem,

$$\begin{aligned}
G(u(k+h), u(k+h-y)) - G(u(k), u(k-y)) &= \Phi_y(k+h) - \Phi_y(k) \\
= h\Phi_y'(k_{y,h}) = h\, \frac{\partial}{\partial k} G(u(k), u(k-y))\Big|_{k_{y,h}}
\end{aligned} \tag{2.28}$$

for some $k_{y,h} \in [k, k+h]$. It follows, via the same argument as above, that

$$\int_{-|h|}^{|h|} \frac{|G(u(k+h), u(k+h-y)) - G(u(k), u(k-y))|}{\sin^2 \frac{y}{2}} \, dy \leq C|h|^{1+\min\{\alpha,\beta\}}$$

also.

Consider the first term on the right hand side of (2.25). By (2.28) we have that for $y \in Y$

$$\begin{aligned}
&G(u(k+h), u(k+h-y)) - G(u(k), u(k-y)) - h\frac{\partial}{\partial k}G(u(k), u(k-y)) \\
&= h\left\{ \frac{\partial}{\partial k}G(u(k), u(k-y))\bigg|_{k_{y,h}} - \frac{\partial}{\partial k}G(u(k), u(k-y))\bigg|_{k} \right\} \\
&= h\{[f(u(k_{y,h})) - f(u(k))] - [f(u(k_{y,h}-y)) - f(u(k-y))]\}u'(k_{y,h}-y) \\
&\quad +\{f(u(k)) - f(u(k-y))\}\{u'(k_{y,h}-y) - u'(k-y)\} \\
&\quad +\{[u(k_{y,h}-y) - u(k-y)] - [u(k_{y,h}) - u(k)]\}f'(u(k_{y,h}))u'(k_{y,h}) \\
&\quad +\{u(k-y) - u(k)\}\{f'(u(k_{y,h}))u'(k_{y,h}) - f'(u(k))u'(k)\};
\end{aligned}$$

we bound each of these terms in turn. For $y \in Y$,

$$\begin{aligned}
&|\{f(u(k_{y,h})) - f(u(k))\} - \{f(u(k_{y,h}-y)) - f(u(k-y))\}||u'(k_{y,h}-y)| \\
&\leq \|u'\|_{L^\infty(E)} \int_{-y}^{0} |f'(u(t+k_{y,h}))u'(t+k_{y,h}) - f'(u(t+k))u'(t+k)| \, dt \\
&\leq C|h|^{\min\{\alpha,\beta\}}|y|
\end{aligned}$$

by Theorem 1.4 (iv), since $f \in C^{1,\beta}(J)$ and u is $C^{1,\alpha}$ on $[-|y| - |h| - |k|, |y| + |h| + |k|] \subset [-\frac{5\delta}{3}, \frac{5\delta}{3}] \subset E$. By Theorem 1.4 (iii), for $y \in Y$,

$$|f(u(k)) - f(u(k-y))||u'(k_{y,h}-y) - u'(k-y)| \leq C|y||k_{y,h} - k|^\alpha \leq C|h|^\alpha |y|,$$

since u is $C^{1,\alpha}$ on $[-|y|-|h|-|k|,|y|+|h|+|k|]\subset[-\frac{5\delta}{3},\frac{5\delta}{3}]\subset E$, and similarly

$$\begin{aligned}
&|\{u(k_{y,h}-y)-u(k-y)\}-\{u(k_{y,h})-u(k)\}||f'(u(k_{y,h}))u'(k_{y,h})|\\
&\leq C\int_{-y}^{0}|u'(t+k_{y,h})-u'(t+k)|\ \mathrm{d}t\\
&\leq C|k_{y,h}-k|^{\alpha}|y|\leq C|h|^{\alpha}|y|.
\end{aligned}$$

Finally, by Theorem 1.4 (iv), for $y\in Y$,

$$\begin{aligned}
|u(k-y)-u(k)||f'(u(k_{y,h}))u'(k_{y,h})-f'(u(k))u'(k)| &\leq C|k_{y,h}-k|^{\min\{\alpha,\beta\}}|y|\\
&\leq C|h|^{\min\{\alpha,\beta\}}|y|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_Y\frac{|G(u(k+h),u(k+h-y))-G(u(k),u(k-y))-h\frac{\partial}{\partial k}G(u(k),u(k-y))|}{\sin^2\frac{y}{2}}\ \mathrm{d}y\\
&\leq C|h|^{1+\min\{\alpha,\beta\}}\int_Y\frac{|y|}{\sin^2\frac{y}{2}}\ \mathrm{d}y\leq C|h|^{1+\min\{\alpha,\beta\}}\int_Y\frac{\mathrm{d}y}{|y|}\leq C|h|^{1+\min\{\alpha,\beta\}}\log\left|\frac{\pi}{2h}\right|;
\end{aligned}$$

so for all h and k in a neighbourhood of 0 we have that

$$\begin{aligned}
&\left|D(k+h)-D(k)-h\int_{-\delta}^{\delta}\frac{\frac{\partial}{\partial k}G(u(k),u(k-y))}{\sin^2\frac{y}{2}}\ \mathrm{d}y\right|\\
&\leq C|h|^{1+\min\{\alpha,\beta\}}+C|h|^{1+\min\{\alpha,\beta\}}\log\left|\frac{\pi}{2h}\right|\leq C|h|^{1+\min\{\alpha,\beta\}}|\log|Ch||.
\end{aligned}$$

(2.23) is thus proved; and the required Hölder continuity follows since D' is uniformly bounded in a neighbourhood of 0, as may be seen from (2.26-2.27).

We now consider the regularity of S. As in the proof of Lemma 2.5, $S(t)$ is given by (2.16); by similar arguments to those used in the same proof, S_1 and the function multiplying the integral in S_2 are in $C^{1,\min\{\alpha,\beta\}}(E)$. We show that the integral in S_2 is in $C^2(\tilde{E})$:

Since $X = S^1 \setminus [-\delta, \delta] = (\delta, 2\pi - \delta)$, we have that for $x \in S^1$

$$\int_X \frac{u(x-y)}{\sin^2 \frac{y}{2}} \, \mathrm{d}y = -\int_{x-\delta}^{x-2\pi+\delta} \frac{u(t)}{\sin^2 \frac{x-t}{2}} \, \mathrm{d}t = -F(x-\delta, x-2\pi+\delta, x),$$

where

$$F(p,q,r) = \int_p^q u(t) \mathrm{cosec}^2 \frac{r-t}{2} \, \mathrm{d}t.$$

Let $K(x) = (x-\delta, x-2\pi+\delta, x) = (K_1(x), K_2(x), K_3(x))$. Then $\int_X \frac{u(x-y)}{\sin^2 \frac{y}{2}} \, \mathrm{d}y = -F(K(x))$. For $x \in \tilde{E}$,

$$\frac{\partial}{\partial x} \int_X \frac{u(x-y)}{\sin^2 \frac{y}{2}} \, \mathrm{d}y = -\left\{ \frac{\partial F}{\partial K_1} \frac{\partial K_1}{\partial x} + \frac{\partial F}{\partial K_2} \frac{\partial K_2}{\partial x} + \frac{\partial F}{\partial K_3} \frac{\partial K_3}{\partial x} \right\}$$

$$= u(x-\delta) \mathrm{cosec}^2 \frac{\delta}{2} - u(x+\delta-2\pi) \mathrm{cosec}^2 \frac{2\pi-\delta}{2} - \int_{x-\delta}^{x-2\pi+\delta} u(t) \frac{\partial}{\partial x} \mathrm{cosec}^2 \frac{x-t}{2} \, \mathrm{d}t. \tag{2.29}$$

Now for $x \in \tilde{E}$, u is $C^{1,\alpha}$ in neighbourhoods of $x-\delta$ and $x-2\pi+\delta$; so the first two terms on the right-hand side of (2.29) are in $C^{1,\alpha}(\tilde{E})$. The third term is

$$-\int_{x-\delta}^{x-2\pi+\delta} u(t) \frac{\partial}{\partial x} \mathrm{cosec}^2 \frac{x-t}{2} \, \mathrm{d}t = \int_X u(x-y) \mathrm{cosec}^2 \frac{y}{2} \cot \frac{y}{2} \, \mathrm{d}y;$$

so by an argument similar to (2.17) and the remarks following, it is in $C^1_{2\pi}$. It follows that the derivative with respect to x of the integral in S_2 is $C^1(\tilde{E})$, so the integral itself is $C^2(\tilde{E})$. Hence, by Theorem 1.4 (iv), $S_2 \in C^{1,\min\{\alpha,\beta\}}(\tilde{E})$.

By a similar argument, $S_3 \in C^{1,\min\{\alpha,\beta\}}(\tilde{E})$.

It follows finally that $S \in C^{1,\min\{\alpha,\beta\}}(\tilde{E})$, and in particular, S is $C^{1,\min\{\alpha,\beta\}}$ in a neighbourhood of 0; together with the regularity of D, this establishes the lemma. $\square$

Theorem 2.9. *(generalisation of theorem 3.7 in [4]) Suppose $f \in C^0(J) \cap C^{1,\beta}(\hat{J} \setminus \mathcal{N}(f))$, where J, $\hat{J}$ and β are as in Theorem 2.1; and $c \in \mathbb{R}$. If w is as in Theorem 2.1 then $w \in C^{2,\alpha}_{\mathrm{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ for each $\alpha \in (0,\beta)$.*

Proof. Let $E_1, \ldots, E_4$ be as in the proof of Theorem 2.6. We show that $w \in C^{2,\alpha}(E_4)$ for each $\alpha \in (0, \beta)$.

By Theorem 2.6, $w \in C^{1,\alpha}(E_1)$ for each $\alpha \in (0,1)$, since $f \in C^{0,1}(\hat{J})$ and $E_1 \subset w^{-1}(\hat{J} \setminus \mathcal{N}(f(w)))$. Hence by Lemma 2.8, for each $\alpha \in (0,1)$ and each $\varepsilon \in (0, \min\{\alpha, \beta\})$, $\mathcal{F}(w) \in C^{1,\varepsilon}(E_2)$: i.e. for each $\alpha \in (0,\beta)$, $\mathcal{F}(w) \in C^{1,\alpha}(E_2)$. By Theorem 1.4 (iv), $f(w) \in C^{1,\alpha}(E_2)$ for each $\alpha \in (0,\beta)$. Since $f(w)$ is bounded away from 0 on E_2, we may apply Theorem 1.4 (ii) to (2.19), yielding $\mathcal{C}w' \in C^{1,\alpha}(E_2)$ for each $\alpha \in (0,\beta)$. Now $\mathcal{C}w'$ is given by (2.20), where $\xi = \xi_{E_2,E_3}$ is as given below (2.10). As in the proof of Theorem 2.6 we have that for each $\alpha \in (0,\beta)$, $\mathcal{C}(\xi w') \in C^{1,\alpha}(E_3) \cap L^1_{2\pi}$. Hence by Theorem 1.8 (2), $\xi w' = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(t) w'(t)\, \mathrm{d}t - \mathcal{C}^2(\xi w') \in C^{1,\alpha}(E_4)$ for each $\alpha \in (0,\beta)$. The result follows since on E_4, $w' \equiv \xi w'$. $\square$

That, under the hypotheses of Lemma 2.8, we may conclude only $\mathcal{F}(u) \in C^{1,\varepsilon}_{\mathrm{loc}}(E^\circ)$ for each $\varepsilon < \min\{\alpha, \beta\}$ rather than $\mathcal{F}(u) \in C^{1,\min\{\alpha,\beta\}}_{\mathrm{loc}}(E^\circ)$, is somewhat unsatisfactory, since it weakens the conclusions of Theorem 2.9 (and Theorem 2.1). However, as the following example shows, Lemma 2.8 is best possible.

Example 2.10. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$ and let $u : S^1 \to \mathbb{R}$ be given by

$$u(x) = \frac{x}{\alpha+1}(|x|^\alpha - \pi^\alpha),$$

where $0 < \alpha \le 1$. Then $f \in C^{1,1}(\mathbb{R})$ (trivially) and $u \in C^{1,\alpha}_{2\pi}$. It follows from Lemma 2.8 that $\mathcal{F}(u)$ is differentiable and (2.22) holds. Let $0 < x < \frac{\pi}{3}$. Lemma 3.6 in [4] shows that

$$\begin{aligned}&(\mathcal{F}(u))'(x) - (\mathcal{F}(u))'(0)\\ &= I_0 + C\int_K \frac{(u'(x)-u'(x-y))(u(x)-u(x-y)) - (u'(0)-u'(-y))(u(0)-u(-y))}{\sin^2 \frac{y}{2}} \mathrm{d}y,\end{aligned}$$

where $K = S^1 \setminus [-2x, 2x]$ and $|I_0| \le Cx^\alpha$ for some constant C, independent of x.

Now

$$\begin{aligned}
&\int_K \frac{[u'(x)-u'(x-y)][u(x)-u(x-y)]-[u'(0)-u'(-y)][u(0)-u(-y)]}{\sin^2\frac{y}{2}}\,\mathrm{d}y \\
&= \left(\int_{-\pi}^{x-\pi}+\int_{\pi-x}^{\pi}\right) \frac{[u'(x)-u'(x-y)]-[u'(0)-u'(-y)]}{\sin^2\frac{y}{2}}[u(x)-u(x-y)]\,\mathrm{d}y \\
&+ \left(\int_{x-\pi}^{-2x}+\int_{2x}^{\pi-x}\right) \frac{[u'(x)-u'(x-y)]-[u'(0)-u'(-y)]}{\sin^2\frac{y}{2}}(u(x)-u(x-y))\,\mathrm{d}y \\
&+ \int_K \frac{[u(x)-u(x-y))-(u(0)-u(-y))][u'(0)-u'(-y)]}{\sin^2\frac{y}{2}}\,\mathrm{d}y \\
&= I_1+I_2+I_3.
\end{aligned}$$

We investigate first the size of I_3. We have that for some constant C, independent of x and y,

$$|(u(x)-u(x-y))-(u(0)-u(-y))| = \left|\int_{-y}^{0} u'(t+x)-u'(t)\,\mathrm{d}t\right| \leq Cx^\alpha|y|,$$

since $u' \in C^{0,\alpha}_{2\pi}$. Hence

$$|I_3| \leq Cx^\alpha \int_K |y|^{\alpha-1}\,\mathrm{d}y \leq Cx^\alpha.$$

Next, we examine the size of I_1.

$$|I_1| \leq Cx^\alpha \left(\int_{-\pi}^{x-\pi}+\int_{\pi-x}^{\pi}\right)\frac{\mathrm{d}y}{|y|} \leq Cx^\alpha \log\frac{\pi}{\pi-x} \leq Cx^\alpha,$$

since $u' \in C^{0,\alpha}_{2\pi}$, $u \in C^{1,\alpha}_{2\pi} \subset C^{0,1}_{2\pi}$ and $\left|\log\left(\frac{\pi}{\pi-x}\right)\right| \leq \log\left(\frac{3}{2}\right)$. Finally, we evaluate

I_2.

$$\begin{aligned}
I_2 &= \int_{2x}^{\pi-x} \frac{(x^\alpha-(y-x)^\alpha+y^\alpha)(x^{\alpha+1}+(y-x)^{\alpha+1}-\pi^\alpha y)}{(\alpha+1)\sin^2\frac{y}{2}}\,\mathrm{d}y \\
&\quad+\frac{(x^\alpha-(y+x)^\alpha+y^\alpha)(x^{\alpha+1}-(y+x)^{\alpha+1}-\pi^\alpha y)}{(\alpha+1)\sin^2\frac{y}{2}}\,\mathrm{d}y. \\
&= \kappa(x)\int_{2x}^{\pi-x} \frac{(x^\alpha-(y-x)^\alpha+y^\alpha)(x^{\alpha+1}+(y-x)^{\alpha+1}-\pi^\alpha y)}{y^2}\,\mathrm{d}y \\
&\quad+\frac{(x^\alpha-(y+x)^\alpha+y^\alpha)(x^{\alpha+1}-(y+x)^{\alpha+1}-\pi^\alpha y)}{y^2}\,\mathrm{d}y,
\end{aligned}$$

for some $\kappa(x)\in\left[\frac{4}{\alpha+1},\frac{\pi^2}{\alpha+1}\right]$, since for $y\in S^1$, $\frac{|y|}{\pi}\le|\sin\frac{y}{2}|\le\frac{|y|}{2}$. Let $t=\frac{y}{x}$. Then

$$\begin{aligned}
I_2 &= \kappa(x)\int_2^{\frac{\pi}{x}-1}\frac{x^\alpha}{t^2}\left\{(1-(t-1)^\alpha+t^\alpha)(x^\alpha+(t-1)^{\alpha+1}x^\alpha-\pi^\alpha t)\right. \\
&\quad\left.+(1-(t+1)^\alpha+t^\alpha)(x^\alpha-(t+1)^{\alpha+1}x^\alpha-\pi^\alpha t)\right\}\,\mathrm{d}t \\
&= \kappa(x)x^\alpha\int_2^{\frac{\pi}{x}-1}\frac{1}{t^2}\left\{(1-t^\alpha[1+\varphi_1(-1/t)]+t^\alpha)(x^\alpha+t^{\alpha+1}[1+\varphi_2(-1/t)]x^\alpha-\pi^\alpha t)\right. \\
&\quad\left.+(1-t^\alpha[1+\varphi_1(1/t)]+t^\alpha)(x^\alpha-t^{\alpha+1}[1+\varphi_2(1/t)]x^\alpha-\pi^\alpha t)\right\}\,\mathrm{d}t,
\end{aligned}$$

where $\varphi_1(s)$ and $\varphi_2(s)$ are the 0th order Taylor series remainders of the functions $s\mapsto(1+s)^\alpha$ and $s\mapsto(1+s)^{\alpha+1}$ about 0 respectively. We have by (1.19) that $|\varphi_i(s)|=O(|s|)$ as $s\to 0$. Hence

$$\begin{aligned}
I_2 &= \kappa(x)x^\alpha\int_2^{\frac{\pi}{x}-1} 2x^\alpha t^{-2}+[\varphi_2(-1/t)-\varphi_2(1/t)]t^{\alpha-1}x^\alpha-2\pi^\alpha t^{-1} \\
&\quad-[\varphi_1(-1/t)+\varphi_1(1/t)]t^{\alpha-2}x^\alpha-[\varphi_1(-1/t)-\varphi_1(1/t)]t^{2\alpha-1}x^\alpha \\
&\quad-[\varphi_1(-1/t)\varphi_2(-1/t)-\varphi_1(1/t)\varphi_2(1/t)]t^{2\alpha-1}x^\alpha \\
&\quad+[\varphi_1(-1/t)+\varphi_1(1/t)]\pi^\alpha t^{\alpha-2}\,\mathrm{d}t. \qquad (2.30)
\end{aligned}$$

Each term on the right-hand side of (2.30) except

$$\kappa(x)x^\alpha\int_2^{\frac{\pi}{x}-1}-2\pi^\alpha t^{-1}\,\mathrm{d}t$$

is dominated by $C\kappa(x)x^\alpha$. The latter term equals

$$2\kappa(x)x^\alpha \log\left(\frac{\pi}{2x}-\frac{1}{2}\right).$$

Hence if $0 < x < \frac{\pi}{3}$ then $(\mathcal{F}(u))'(x) - (\mathcal{F}(u))'(0) = R_x + 2\kappa(x)x^\alpha \log\left(\frac{\pi}{2x}-\frac{1}{2}\right)$, where $|R_x| \le C|x|^\alpha$ and $\kappa(x)$ is bounded and bounded away from 0 as $x \to 0$. Since, as $x \to 0$, $\log\left(\frac{\pi}{2x}-\frac{1}{2}\right) \to \infty$, it follows that $\mathcal{F}(u) \notin C^{1,\alpha}_{2\pi}$. ■

This concludes the case $n = 1$. Before we can prove the inductive step in the proof of Theorem 2.1 (I), we require the following two lemmas:

Lemma 2.11. *Suppose $f \in C^0(J)) \cap C^{n,\beta}(\hat{J})$ where $n \in \mathbb{N}$ and β, J and $\hat{J}$ are as in Theorem 2.1. If $u \in W^{1,1}_{2\pi} \cap C^{n,\alpha}(E)$ with $\alpha \in (0,1]$, $E \subset u^{-1}(\hat{J})$ and $\mathcal{R}(u) \subset J$, then $\mathcal{F}(u)$ is $(n-1)$-times differentiable on E°, and for x in each compact segment, $\tilde{E}$ of E°,*

$$(\mathcal{F}(u))^{(n-1)}(x) = f(u(x))\mathcal{C}(\xi u^{(n)})(x) - \mathcal{C}\left(f(u)\xi u^{(n)}\right)(x) + R_{n-1}(u)(x),$$

where ξ is a smooth function on S^1 and $R_{n-1}(u) \in C^{1,\min\{\alpha,\beta\}}(\tilde{E})$.

Proof. Let $\tilde{E}$ be a compact segment of E°, and let $\hat{E} = \mathcal{E}(E, \tilde{E})$ be as given in (2.10). Note that $\tilde{E} \subset \hat{E}^\circ$. Let also $\xi = \xi_{E,\tilde{E}}$ be as given below (2.10). Then we have

$$\begin{aligned}\mathcal{F}(u) &= f(u)\mathcal{C}u' - \mathcal{C}(f(u)u') \\ &= f(u)\mathcal{C}(\xi u') - \mathcal{C}(f(u)\xi u') + f(u)\mathcal{C}((1-\xi)u') - \mathcal{C}(f(u)(1-\xi)u').\end{aligned}$$

By the proof of Theorem 1.8, $\mathcal{C}((1-\xi)u')$ and $\mathcal{C}(f(u)(1-\xi)u')$ are smooth on $\tilde{E}$. We consider

$$\mathcal{H}(\xi, u) := f(u)\mathcal{C}(\xi u') - \mathcal{C}(f(u)\xi u').$$

Note that $f(u)\xi u', \xi u' \in C^{n,\alpha}_{2\pi}$, so in differentiating $\mathcal{H}(\xi, u)$ $n-1$ times, we may interchange the operators $\mathcal{C}$ and differentiation. By Leibniz' rule and the Fàa-de Bruno formula (1.8), we have that on E,

$$\begin{aligned}
&(\mathcal{H}(\xi,u))^{(n-1)}(x) \\
&= f(u)\mathcal{C}v^{(n-1)} - \mathcal{C}\left(f(u)v^{(n-1)}\right) \\
&\quad + \sum_{k=1}^{n-1}\binom{n-1}{k}\left\{\mathcal{C}v^{(n-1-k)}(x)\frac{\mathrm{d}^k}{\mathrm{d}x^k}f(u(x)) - \mathcal{C}\left(v^{(n-1-k)}\frac{\mathrm{d}^k}{\mathrm{d}x^k}f(u)\right)(x)\right\} \\
&= f(u)\mathcal{C}v^{(n-1)} - \mathcal{C}\left(f(u)v^{(n-1)}\right) \\
&\quad + \sum_{k=1}^{n-1}\binom{n-1}{k}\left[\sum^{k}\frac{k!}{\prod_{j=1}^{m} i_j!}\left\{f^{(\sum_{j=1}^{m} i_j)}(u)\prod_{j=1}^{m}\left(\frac{u^{(j)}}{j!}\right)^{i_j}\mathcal{C}v^{(n-1-k)}\right.\right. \\
&\qquad\qquad \left.\left. - \mathcal{C}\left(f^{(\sum_{j=1}^{m} i_j)}(u)\prod_{j=1}^{m}\left(\frac{u^{(j)}}{j!}\right)^{i_j}v^{(n-1-k)}\right)\right\}\right], \qquad (2.31)
\end{aligned}$$

where $v = \xi u'$. All the terms on the right-hand side of (2.31) except $f(u)\mathcal{C}v^{(n-1)} - \mathcal{C}(f(u)v^{(n-1)})$ involve only derivatives of f of order less than or equal to $n-1$ and derivatives of u of order less than or equal to $n-1$; so Theorem 1.8 (2), together with Theorem 1.4 (iii) and (iv) shows that their sum is in $C^{1,\min\{\alpha,\beta\}}(\tilde{E})$.

Now by Leibniz' rule,

$$v^{(n-1)} = \sum_{i=0}^{n-1}\binom{n-1}{i}\xi^{(i)}u^{(n-i)}, \qquad (2.32)$$

and all terms on the right-hand side of (2.32) except $\xi u^{(n)}$ involve only derivatives of u of order less than or equal to $n-1$, so are in $C^{1,\alpha}(\tilde{E})$. It follows that all terms except

$$f(u)\mathcal{C}\left(\xi u^{(n)}\right) - \mathcal{C}\left(f(u)\xi u^{(n)}\right)$$

in the $n-1$th derivative of $\mathcal{F}(u)$ are in $C^{1,\min\{\alpha,\beta\}}(\tilde{E})$, as required. □

Lemma 2.12. *Suppose $a \in W^{1,1}_{2\pi} \cap C^{2,\alpha}(E)$, where $\alpha \in (0,1]$ and $E \subset S^1$; and*

$b \in L^1_{2\pi}$. $\mathcal{G}(a,b)$, *given by*

$$\mathcal{G}(a,b)(x) = a(x)\mathcal{C}b(x) - \mathcal{C}(ab)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\{a(x)-a(y)\}b(y)}{\tan\frac{x-y}{2}}\,\mathrm{d}y,$$

is in $C^{1,\alpha}_{\mathrm{loc}}(E^\circ)$.

Proof. Choose a compact segment $\tilde{E}$ of E°, let $\delta = \delta(\tilde{E}, E)$ and $\hat{E} = \mathcal{E}(\tilde{E}, E)$ be as in (2.9-2.10), and let $y \in S^1$ be fixed. Let

$$A(x,y) = \begin{cases} \dfrac{a(x)-a(y)}{\tan\frac{x-y}{2}} & x \in S^1 \setminus \{y\} \\ 2a'(y) & x = y \end{cases}$$

We shall show that $A(\cdot, y) \in C^{1,\alpha}(\hat{E})$.

If $y \in S^1 \setminus \hat{E}$, then clearly $A(\cdot,y) \in C^{2,\alpha}(\hat{E})$, by Theorem 1.4 (ii). Otherwise, $A(\cdot,y) \in C^{2,\alpha}(\hat{E}\setminus[y-\frac{\delta}{2}, y+\frac{\delta}{2}])$ by Theorem 1.4 (ii). If $y \in \hat{E}$ and $x \in [y-\frac{\delta}{2}, y+\frac{\delta}{2}]$, then

$$A(x,y) = \{a(x)-a(y)\}B(x,y) + 2\int_0^1 a'(y+t[x-y])\,\mathrm{d}t,$$

where

$$B(x,y) = \begin{cases} \dfrac{1}{\tan\frac{x-y}{2}} - \dfrac{2}{x-y} & x \neq y \\ 0 & x = y. \end{cases}$$

This follows since if $x = y$ then $\int_0^1 a'(y+t[x-y])\,\mathrm{d}t = a'(y)$, whilst if $x \neq y$, then

$$\int_0^1 a'(y+t[x-y])\,\mathrm{d}t = \frac{1}{x-y}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}a(y+t[x-y])\,\mathrm{d}t = \frac{a(x)-a(y)}{x-y}.$$

$B(\cdot,y)$ is infinitely differentiable on $[y-\frac{\pi}{2}, y+\frac{\pi}{2}]$, and it follows from the Hölder continuity of a'' that $A(\cdot,y) \in C^{1,\alpha}([y-\frac{\delta}{2}, y+\frac{\delta}{2}])$; so $A(\cdot,y) \in C^{1,\alpha}(\hat{E})$, as required.

Since for each $y \in S^1$, $A(\cdot,y) \in C^{1,\alpha}(\hat{E})$, it follows that $A(\cdot,y) \in C^{1,\alpha}(\hat{E})$ uniformly in y.

We now show that $\mathcal{G}(a,b)$ is differentiable on $\tilde{E}$, and that for $x \in \tilde{E}$,

$$(\mathcal{G}(a,b))'(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A^{(1,0)}(x,y)b(y)\;\mathrm{d}y. \tag{2.33}$$

For fixed $x \in \tilde{E}$ and for $|h| < \delta$, we have

$$\begin{aligned}\mathcal{G}(a,b)(x+h) - \mathcal{G}(a,b)(x) - \frac{h}{2\pi}\int_{-\pi}^{\pi} A^{(1,0)}(x,y)b(y)\;\mathrm{d}y \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi}\{A(x+h,y) - A(x,y) - hA^{(1,0)}(x,y)\}b(y)\;\mathrm{d}y.\end{aligned}$$

Now by the mean value theorem, for each $x \in \tilde{E}$ and $y \in S^1$, $A(x+h,y) - A(x,y) = hA^{(1,0)}(x_{y,h},y)$ for some $x_{y,h} \in [x, x+h]$. It follows that

$$|A(x+h,y) - A(x,y) - hA^{(1,0)}(x,y)| = |h||A^{(1,0)}(x_{y,h},y) - A^{(1,0)}(x,y)| \le C|h|^{1+\alpha},$$

for some constant C, independent of x, y and h; and (2.33) follows, as required. That $\mathcal{G}(a,b) \in C^{1,\alpha}(\tilde{E})$ may now be seen from a calculation similar to (2.24). □

Corollary 2.13. *Suppose $f \in C^0(J) \cap C^{n,\beta}(\hat{J})$, where $n \in \mathbb{N} \setminus \{1\}$ and β, J and $\hat{J}$ are as in Theorem 2.1. If $u \in W^{1,1}_{2\pi} \cap C^{n,\alpha}(E)$ with $\alpha \in (0,1]$, $E \subset u^{-1}(\hat{J})$ and $\mathcal{R}(u) \subset J$ then $\mathcal{F}(u) \in C^{n,\min\{\alpha,\beta\}}_{\mathrm{loc}}(E^{\circ})$.*

Proof. Let $\tilde{E}$ be a compact segment of E°. By Lemma 2.11, it will suffice to show that $\mathcal{H}_{n-1}(\xi,u) := f(u)\mathcal{C}(\xi u^{(n)}) - \mathcal{C}(f(u)\xi u^{(n)}) \in C^{1,\min\{\alpha,\beta\}}(\tilde{E})$, where ξ is as in Lemma 2.11. This follows from Lemma 2.12, since $f(u) \in C^{n,\min\{\alpha,\beta\}}(\tilde{E})$ (by Theorem 1.4 (iv)), $\xi u^{(n)} \in L^1_{2\pi}$ (trivially) and $\mathcal{H}_{n-1}(\xi,u) = \mathcal{G}(f(u), \xi u^{(n)})$. □

Proof of Theorem 2.1 (I). The cases $n = 0$ and $n = 1$ are proved in Theorems 2.6 and 2.9 respectively. Assume that (2.1) holds if $n = m \in \mathbb{N}$; we show that (2.2) holds if $\beta < 1$ and $n = m+1$, and that (2.1) holds if $\beta = 1$ and $n = m+1$.

Let $f \in C^{m+1,\beta}(\hat{J} \setminus \mathcal{N}(f))$, let $w \in W^{1,1}_{2\pi}$ be a solution of (1.1) such that $\mathcal{C}w' \in L^1_{\mathrm{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ and let $E_1, \dots, E_4$ be as in the proof of Theorem 2.6.

Let first $\beta < 1$. We show that $w \in C^{m+2,\beta}(E_4)$. By hypothesis, $w \in C^{m+1,\alpha}(E_1)$ for all $\alpha \in (0,1)$, since $f \in C^{m,1}(\hat{J} \setminus \mathcal{N}(f))$ and $E_1 \subset w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$. Hence by Corollary 2.13, $\mathcal{F}(w) \in C^{m+1,\beta}(E_2)$. Also $f(w) \in C^{m+1,\beta}(E_2)$ by Theorem 1.4 (iv), since $f \in C^{m+1,\beta}(\hat{J} \setminus \mathcal{N}(f))$, $w \in C^{m+1,\beta}(E_2)$ and $E_2 \subset w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$.

Since $f(w)$ is bounded away from 0 on E_2, we may apply Theorem 1.4 (ii) to (2.19), and conclude that $\mathcal{C}w' \in C^{m+1,\beta}(E_2)$. Now $\mathcal{C}w'$ is given by (2.20), where $\xi = \xi_{E_2,E_3}$ is as given below (2.10). As in the proof of Theorem 2.6 we have that $\mathcal{C}(\xi w') \in C^{m+1,\beta}(E_3) \cap L^1_{2\pi}$. Hence by Theorem 1.8 (2) , $\xi w' = \frac{1}{2\pi}\int_{-\pi}^{\pi} \xi(t)w'(t)\ \mathrm{d}t - \mathcal{C}^2(\xi w') \in C^{m+1,\beta}(E_4)$. Hence $w \in C^{m+2,\beta}(E_4)$, since on E_4, $w' \equiv \xi w'$.

In the case $\beta = 1$, let $\alpha \in (0,1)$. We show that $w \in C^{m+2,\alpha}(E_4)$. As before, by hypothesis, $w \in C^{m+1,\alpha}(E_1)$. Hence, similarly to above, $\mathcal{F}(w) \in C^{m+1,\alpha}(E_2)$ and $f(w) \in C^{m+1,\alpha}(E_2)$.

We apply again Theorem 1.4 (ii) to (2.19), and conclude that $\mathcal{C}w' \in C^{m+1,\alpha}(E_2)$; and as above $\xi w' \in C^{m+1,\alpha}(E_4)$. Hence $w \in C^{m+2,\alpha}(E_4)$, since on E_4, $w' \equiv \xi w'$.

The theorem now follows by induction. $\square$

2.3 Real-analyticity

In this section we present a proof of the second part of Theorem 2.1, and also show that the hypotheses of Theorem 2.1 are satisfied by solutions $w \in W^{1,1}_{2\pi}$ if any of the conditions (A)-(C) are satisfied.

Proof of Theorem 2.1 (II). If $w' \equiv 0$ then the result is trivial, so we may assume that $w'(t) \not\equiv 0$ on a set of positive measure.

Let E_2 be a compact segment of $w^{-1}(\hat{J}) \setminus \mathcal{N}(\mathcal{Z})$, and let E_1 be a compact segment of $w^{-1}(\hat{J}) \setminus \mathcal{N}(\mathcal{Z})$ such that $E_2 \subset E_1^{\circ}$. We show that w is real-analytic on E_2. Let $\mathcal{R} : \{x + iy \in \mathbb{C} | y < 0\} \to \mathbb{C}$ be given by

$$\mathcal{R}(x + iy) = F(\mathrm{e}^{-ix+y}), \tag{2.34}$$

where $F : \mathcal{D} \to \mathbb{C}$ is given by (1.13). Then $\mathcal{R} = \mathcal{U} + i\mathcal{V}$ is holomorphic and as $y \nearrow 0$, $\mathcal{R}(x + iy) \to w(x) - i\mathcal{C}w(x)$ pointwise almost everywhere. By Theorem 2.9, $w, \mathcal{C}w \in C^{2,\alpha}(E_1)$ for each $\alpha \in (0,1)$. Hence by Lemma 1.5, $\mathcal{U}, \mathcal{V} \in C^{2,\alpha}_{\mathrm{loc}}(\{x+$

$iy \in \mathbb{C} | x \in E_1, y \leq 0\}$) (note that $\mathcal{U}$ and $\mathcal{V}$ satisfy the strictly elliptic equation $\Delta u = 0$). Hence for $x \in E_1$,

$$\mathcal{U}_x(x,y) \to w'(x), \quad \mathcal{U}_y(x,y) \to (\mathcal{C}w)'(x). \tag{2.35}$$

as $y \nearrow 0$. It follows that for $x \in E_1$, $\mathcal{R}'(x+iy) \to w'(x) - i(\mathcal{C}w)'(x)$ as $y \nearrow 0$. Now by Theorem 2.1 (I), $w' \in C^{2,\alpha}(E_1)$, so by Theorem 1.8 *(1)*, $\mathcal{C}w' \in C^{2,\alpha}(E_2)$. By the same argument as above, there exists a holomorphic function $\tilde{\mathcal{R}} = \tilde{\mathcal{U}} + \tilde{\mathcal{V}}$ on $\{x+iy \in \mathbb{C} | y < 0\}$, with $C^{2,\alpha}$ extension $w' - i\mathcal{C}w'$ to E_2.

Now for $y < 0$, $\mathcal{U}(x+iy) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{e^{-y}}(x-t)w(t)\ dt$, where for $r < 1$ and $t, \theta \in S^1$, $P_r(\theta - t)$ is the **Poisson kernel**,

$$P_r(\theta - t) = \mathrm{Re}\left[\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}}\right] = \frac{1-r^2}{1 - 2r\cos(\theta - t) + r^2}.$$

Note that $\frac{\partial}{\partial t}P_r(\theta - t) = -\frac{\partial}{\partial \theta}P_r(\theta - t)$. By integrating by parts we obtain

$$\frac{\partial}{\partial x}\mathcal{U}(x+iy) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\partial}{\partial t}P_{e^{-y}}(x-t)\right] w(t)\ dt = \tilde{\mathcal{U}}(x+iy)$$

It follows that $\mathcal{R}'$ and $\tilde{\mathcal{R}}$ are holomorphic functions on the open lower half-plane with equal real parts, so must differ by an imaginary constant. In particular, $\mathcal{C}w'$ and $(\mathcal{C}w)'$ differ by a constant on E_2.

Since $f(w)$ and $\mathcal{Z}$ are bounded away from 0 on E_2, f is real-analytic on $w(E_2)$ and $\mathcal{Z}$ is real-analytic on E_2,

$$\mathcal{C}w'(t) = \sqrt{\frac{\mathcal{Z}(t)}{f(w(t))} - w'(t)^2 - 1}$$

gives $(\mathcal{C}w)'$ as an analytic function of t, w and w' on $E_2 \setminus \mathcal{N}(1 + \mathcal{C}w')$: i.e. it gives $\mathcal{U}_y$ as an analytic function of t, $\mathcal{U}$ and $\mathcal{U}_x$ on $E_2 \setminus \mathcal{N}(1 + \mathcal{C}w')$; and by Lewy's theorem (Theorem 1.13), $\mathcal{R}$ extends to an analytic function on a ball centred at each point of $E_2 \setminus \mathcal{N}(1 + \mathcal{C}w')$. Hence in particular w is real-analytic on $E_2 \setminus \mathcal{N}(1 + \mathcal{C}w')$.

A similar argument, using a different function $\tilde{\mathcal{R}}$ on the lower half-plane and the formula

$$w'(t) = \sqrt{\frac{\mathcal{Z}(t)}{f(w(t))} - (1 + \mathcal{C}w(t))^2}$$

shows that w is real-analytic on $E_2 \setminus \mathcal{N}(w')$. Since on E_2, $w'^2 + (1 + \mathcal{C}w')^2 \neq 0$, we have that w is real-analytic on E_2. $\square$

Theorem 2.14. *Suppose f, c and w satisfy the hypotheses of Theorem 2.1 (II). Then w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$.*

Proof. If $f(w) \equiv 0$ then the result is trivial; so assume $f(w) \not\equiv 0$. By the proof of Theorem 2.3 (2), $\Phi := HW$ (where $H = \mathcal{O}(\sqrt{|f(w)|})$ and $W \in \cap_{p<1}\mathcal{H}^p_{\mathbb{C}}$ is such that $W^* = w' + i(1 + \mathcal{C}w')$) has an analytic extension to $(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))^* \cup (\mathbb{C} \setminus \partial\mathcal{D})$. It follows by (1.17) that on $S^1 \setminus \mathcal{N}(f(w))$,

$$W^* = \left(\frac{\Phi}{H}\right)^* = \frac{\Phi^*}{\mathcal{O}(\sqrt{|f(w)|})^*} = \frac{\Phi^*}{\sqrt{|f(w)|}} \exp\{-i\mathcal{C}(\log \sqrt{|f(w)|})\}. \tag{2.36}$$

Let $g = \log \sqrt{|f(w)|}$. Then by (2.36) we have

$$w' = \frac{1}{\mathrm{e}^g}(\mathrm{Re}\Phi^* \cos(\mathcal{C}g) + \mathrm{Im}\Phi^* \sin(\mathcal{C}g)).$$

Now $g' = f'(w)w'/(2f(w))$, so

$$g' = \frac{f'(w)}{2\sigma \mathrm{e}^{3g}}(\mathrm{Re}\Phi^* \cos(\mathcal{C}g) + \mathrm{Im}\Phi^* \sin(\mathcal{C}g)),$$

where $\sigma = \mathrm{sign}(f(w))$ is constant (± 1) on each compact segment of $S^1 \setminus \mathcal{N}(f(w))$. Finally, if $f'(w(t)) \neq 0$ then in a neighbourhood of $w(t)$, f has a local analytic inverse, $f^{-1}_{w(t)}$. Hence in a neighbourhood of t, $w(s) = f^{-1}_{w(t)}(\sigma \mathrm{e}^{2g(s)})$. It follows that in a neighbourhood of t,

$$g'(s) = \frac{f'(f^{-1}_{w(t)}(\sigma \mathrm{e}^{2g(s)}))}{\sigma \mathrm{e}^{3g(s)}}(\mathrm{Re}\Phi^*(s) \cos(\mathcal{C}g(s)) + \mathrm{Im}\Phi^*(s) \sin(\mathcal{C}g(s))). \tag{2.37}$$

Let $t \in w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$, and let E_1, E_2 and E_3 be compact segments of $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$ such that $t \in E_3$, (2.37) holds for all $t \in E_3$, $E_3 \subset E_2^\circ$ and $E_2 \subset E_1^\circ$. We show that w is real-analytic on E_3.

By Theorem 2.9, $w \in C^{2,\alpha}(E_1)$, so by Theorem 1.4 (iv), $g \in C^{2,\alpha}(E_1)$ (note that $f \neq 0$ on $w(E_1)$, so on $f(w)(E_1)$, $t \mapsto |t|$ is smooth; and on $|f(w)|(E_1)$, log is smooth). Now $f(w) \in \mathrm{Log}$ (see the proof of Theorem 2.3), so $g \in L^1_{2\pi}$; hence by Theorem 1.8 (2), $\mathcal{C}g \in C^{2,\alpha}(E_2)$.

A similar argument to the one in (2.34-2.35) shows that there exists a function $\mathcal{R} = \mathcal{U} + i\mathcal{V}$, holomorphic on the open lower half plane in $\mathbb{C}$, with boundary data $g - i\mathcal{C}g$ on $\mathbb{R}$. As in the proof of Theorem 2.1 (II), Lemma 1.5 now shows that $\mathcal{U}, \mathcal{V} \in C^{2,\alpha}_{\mathrm{loc}}(\{x+iy \in \mathbb{C} | x \in E_2, y \leq 0\})$, whence by Lewy's theorem (Theorem 1.13) and (2.37), $\mathcal{R}$ extends to an analytic function in a ball about every point in E_3; and in particular $g, \mathcal{C}g$ are real-analytic on E_3.

Now for $t \in E_3$, $w(s) = f^{-1}_{w(t)}(\sigma e^{2g(s)})$ for all s in a neighbourhood of t, so w is real analytic on E_3 also. We have shown that for each $t \in w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$, there is a neighbourhood of t on which w is real-analytic. Hence w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)f'(w))$, as required. $\square$

Corollary 2.15. *If f, c and w satisfy the hypotheses of Theorem 2.1 (II) and $f' \neq 0$ on $\mathcal{R}(w)$ then w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$.*

We now show that the hypotheses of Theorem 2.1 are satisfied by solutions $w \in W^{1,1}_{2\pi}$ of (1.1) in cases (A)-(C).

In case (A), $w \in W^{1,p}_{\mathrm{loc}}(w^{-1}(\hat{J}))$ for some $p > 1$, so by Theorem 1.8 (1), $\mathcal{C}w' \in L^p_{\mathrm{loc}}(w^{-1}(\hat{J})) \subset L^1_{\mathrm{loc}}(w^{-1}(\hat{J}))$. In case (B), $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$, so $\mathcal{C}w' \in L^1_{2\pi}$. For case (C) we have the following:

Lemma 2.16. *Let $u \in W^{1,1}_{2\pi}$ with $\mathcal{R}(u) \subset J$. If $f \in C^0(J)$ is monotone increasing on $\mathcal{R}(u)$ then for almost all $x \in S^1$, $\mathcal{F}(u)(x) \geq 0$. If f is monotone decreasing on $\mathcal{R}(u)$ then $\mathcal{F}(u)(x) \leq 0$ almost everywhere.*

Proof. Let $x, y \in S^1$. By the mean value theorem, $\phi(u(x)) - \phi(u(x-y)) = f(\chi)\{u(x) - u(x-y)\}$ for some $\chi \in (u(x), u(x-y))$. Hence $G(u(x), u(x-y)) = \{f(\chi) - f(u(x))\}\{u(x) - u(x-y)\}$. Suppose f is monotone decreasing. Then $f(\chi) - f(u(x))$ has the same sign as $u(x) - u(x-y)$, so $G(u(x), u(x-y)) \geq 0$. Hence by (2.8), $\mathcal{F}(u)(x) \leq 0$ almost everywhere. Similarly if f is monotone increasing. □

Corollary 2.17. *Suppose $f \in C^0(J)$. If $w \in W^{1,1}_{2\pi}$ is a solution of (1.1) such that f is monotone on $\mathcal{R}(w)$ then $\mathcal{C}w' \in L^1_{\text{loc}}(S^1 \setminus \mathcal{N}(f(w)))$.*

Proof. Since w satisfies (1.1), $f(w) + \mathcal{F}(w) = c - 2\mathcal{C}(f(w)w')$. Hence by Lemma 2.16, $\mathcal{C}(f(w)w')$ is either bounded above or below. Hence by the remark following Definition 1.10, $\mathcal{C}(f(w)w') \in L^1_{2\pi}$. Hence $\mathcal{F}(w) \in L^1_{2\pi}$, so by (2.18), $\mathcal{C}w' \in L^1_{\text{loc}}(S^1 \setminus \mathcal{N}(f(w)))$, as required. □

2.4 The Bernoulli condition

In this section we investigate the case where $\mathcal{Z} \equiv c$. This case is important here because of the following corollary to Theorem 2.1 (II)

Corollary 2.18. *Suppose f and w satisfy the hypotheses of Theorem 2.1 (II), and $c \in \mathbb{R} \setminus \{0\}$. If $\mathcal{Z} \equiv c$ on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ then w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$.*

Proof. By Theorem 2.1 (II), w is real-analytic on $w^{-1}(\hat{J}) \setminus \mathcal{N}(\mathcal{Z})$. On $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$, $\mathcal{Z} \equiv c \neq 0$. The result follows. □

We now consider which solutions $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ of (1.1) satisfy $\mathcal{Z} \equiv c$. The basic result is the following:

Theorem 2.19. *Let $f \in C^0(J)$ and $c \in \mathbb{R}$. Suppose $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$ satisfies (1.1). If*

$c \neq 0$ then the following are equivalent.

(α) *$f(w(t))/c \geq 0$ almost everywhere,* (β) *$\mathcal{Z}(t) = c$ almost everywhere,*

(γ) *$\mathcal{Z} \in L^1_{2\pi}$,* ($\delta$) *$\mathcal{Z}$ is bounded above (or below).*

If $c = 0$ then (β)-(δ) are equivalent, and are implied by

(α') *$f(w(t))$ has the same sign almost everywhere.*

Note that in this case, (β) is equivalent to $f(w(t)) = 0$ almost everywhere.

We include a proof for completeness, although the result is a corollary of theorem 2.1 in [17].

Proof. By the proof of Theorem 2.3, (1.1) becomes the Riemann-Hilbert problem (2.4). In the case $w \in \mathcal{H}^{1,1}_{\mathbb{R}}$, we have that $|W^*| = |w' + i(1 + Cw')| \in L^1_{2\pi}$, so $W \in \mathcal{H}^1_{\mathbb{C}}$. Also, $|U^*| = |f(w)||W^*| \in L^1_{2\pi}$, so $U \in \mathcal{H}^1_{\mathbb{C}}$ by Smirnov's theorem (Theorem 1.12).

Suppose that $c \neq 0$ and (α) holds, or that $c = 0$ and (α') holds. Then, using the notation of the proof of Theorem 2.3, either S^+, or S^- has zero measure. Suppose, without loss of generality that S^- has measure zero. Then $(U/H)^* = \overline{(HW)^*}$ almost everywhere on S^1 and $(U/H)^*, (HW)^* \in L^1_{2\pi}$ (see the proof of Theorem 2.3 for notation). Hence by Theorem 2.2, U/H and HW have bounded analytic extensions to $\mathbb{C}$, so are constant. Hence $\mathcal{Z} = U^*W^* = (U/H)^*(HW)^*$ is constant. Since $W(0) = i$ and $U(0) = -ic$, we have that $\mathcal{Z}(t) = c$ almost everywhere: i.e. (β) holds.

Clearly (β) $\Rightarrow$ (α) if $c \neq 0$; and in all cases (β) $\Rightarrow$ (γ) and (β) $\Rightarrow$ (δ). Suppose (γ) holds. By (2.4),

$$U^*(t)^2 = f(w(t))^2 \overline{W^*(t)^2} \tag{2.38}$$

Let $\tilde{H} = \mathcal{O}(f(w))$, and for $z \in \mathcal{D}$,

$$\tilde{F}(z) = \frac{U(z)^2}{\tilde{H}(z)}, \quad \tilde{G}(z) = \tilde{H}(z)W(z)^2.$$

Now $|\tilde{G}^*(t)| = |f(w(t))||W^*(t)|^2 = |\mathcal{Z}(t)|$, so $\tilde{G} \in \mathcal{H}^1_{\mathbb{C}}$. Also, similarly to (2.5), $|\tilde{F}(z)| \leq |\mathcal{O}(G^*)(z)|$, so $\tilde{F} \in \mathcal{H}^1_{\mathbb{C}}$. By (2.38) and (1.17), we have that for $t \in S^1$,

$$\tilde{F}^*(t) = \frac{U^*(t)^2}{\tilde{H}^*(t)} = \frac{\overline{\tilde{H}^*(t)}U^*(t)^2}{|\tilde{H}^*(t)|^2} = \overline{\tilde{H}^*(t)W^*(t)^2} = \overline{\tilde{G}^*(t)}.$$

Hence by Theorem 2.2, $\tilde{F}$ and $\tilde{G}$ have bounded analytic extensions to $\mathbb{C}$, so are constant. In particular, $(UW)^2 = FG$, and hence UW, is constant, so $\mathcal{Z}$ is almost everywhere constant on S^1. As above we have $\mathcal{Z}(t) = c$ almost everywhere: i.e. (β) holds.

Suppose (δ) holds, and suppose that $\mathcal{Z}(t) \geq d$ for some $d \in \mathbb{R}$. Then $\Phi^*(t) \geq 1$, where, $\Phi = UW - d + 1 \in \mathcal{H}^{\frac{1}{2}}_{\mathbb{C}}$. Let

$$F = \frac{\Phi}{\mathcal{O}(\sqrt{\Phi^*})}, \quad G = \mathcal{O}(\sqrt{\Phi^*}).$$

Then for $t \in S^1$, $F^*(t) = \overline{G^*(t)}$, and F and G are in $\mathcal{H}^1_{\mathbb{C}}$, since G is; so as above F and G are constant. Hence UW is constant, so $\mathcal{Z}$ is almost everywhere constant on S^1. If $\mathcal{Z} \leq d$, then a similar argument shows that the function $d - UW + 1$ is constant, and hence that $\mathcal{Z}$ is almost everywhere constant. As before, we must have $\mathcal{Z}(t) = c$ almost everywhere: i.e. (β) holds.

If $c = 0$ and (β) holds, then $f(w(t))\{w'(t)^2 + (1 + \mathcal{C}w'(t))^2\} = 0$ almost everywhere. Since $w'(t)^2 + (1 + \mathcal{C}w'(t))^2 \neq 0$ almost everywhere (see the proof of Theorem 2.3 (1)), we must have $f(w(t)) = 0$ almost everywhere. $\square$

The final result in this chapter relies on the following result from [17]

Lemma 2.20. *Suppose* $a \in C^{0,\alpha}_{2\pi} \setminus \{0\}$ *with* $a(t_0) = 0$ *for some* $t_0 \in S^1$*; and* $\Phi, \Psi \in \mathcal{H}^p_{\mathbb{C}}$ $(p \geq 1)$*. If* $p \geq 2/\alpha$ *then the equation* $\Psi^* = a\overline{\Phi^*}$ *has no nonconstant solutions.*

Theorem 2.21. *(generalisation of theorem 1.7 (a) and (b) in [17])*

(a) *Let* $f \in C^{0,\beta}(J)$, *where* $\beta \in [0,1]$ *[*$C^{0,0}(J) := C^0(J)$*] and* $J \subset \mathbb{R}$ *is open, and* $c \in \mathbb{R}$. *Let* $w \in W_{2\pi}^{1,\frac{2+\beta}{1+\beta}}$ *be a solution of (1.1). If* $c \neq 0$ *then* (α) *-* (δ) *in Theorem 2.19 hold. If* $c = 0$ *then* (β) *-* (δ) *hold.*

(b) *Let* $f : J \to \mathbb{R}$ *be real-analytic, and* $c \in \mathbb{R}\backslash\{0\}$. *If* $w \in \mathcal{H}_{\mathbb{R}}^{1,1}$ *is a solution of (1.1) then the following are equivalent: (i)* $w \in W_{2\pi}^{1,3}$, *(ii)* w *is real-analytic, (iii)* $f(w)/c > 0$ *everywhere.*

Proof. (a) By Theorem 2.19, we need only show that (γ) holds. If $f \in C^0(J)$ and $w \in W_{2\pi}^{1,2}$, this follows easily from Hölder's inequality and the theorem of M. Riesz (Theorem 1.7). Otherwise, by Lemma B.1, we have that $\mathcal{F}(w) \in L_{2\pi}^{2+\beta}$. Now since $\mathcal{F}(w) = c - f(w) - 2\mathcal{C}(f(w)w') = 2f(w)\mathcal{C}w' + f(w) - c$, and since $f(w) \in L_{2\pi}^{\infty}$, we have that $\mathcal{C}(f(w)w'), f(w)\mathcal{C}w' \in L_{2\pi}^{2+\beta}$. Hence by the theorem of M. Riesz (Theorem 1.7) and by Hölder's inequality, $\mathcal{Z} \in L_{2\pi}^1$, as required.

(b) Suppose (iii) holds. Then by Theorem 2.19, (α) - (γ) hold, and w is real-analytic everywhere by Corollary 2.18. Hence (iii)$\Rightarrow$(ii)$\Rightarrow$(i).

Suppose (i) holds. Then $f(w) \in C_{2\pi}^{0,2/3}$ by Theorem 1.4 (iii) and Hölder's inequality. Now $W \in \mathcal{H}_{\mathbb{C}}^3$, so as $U^* = f(w)\overline{W^*}$, $U \in \mathcal{H}_{\mathbb{C}}^3$. Hence by Lemma 2.20, either U^* and W^* are constant (in which case w is constant and (i)-(iii) hold since $f(w) = c \neq 0$) or $f(w)$ is nowhere 0. By part (a), $\mathcal{Z} = c$, so $f(w)/c > 0$ everywhere: i.e. (iii) holds. $\square$

Chapter 3

Local Bifurcation Theory

In this chapter and the next, we consider one-parameter families of functions $f(\lambda, \cdot)$ where $\lambda \in \mathbb{R}$; and find bifurcation points $(\lambda^*, 0) \in \mathbb{R} \times W^{1,2}_{2\pi}$ of (1.4). We begin with some notation.

Let I and J be open intervals in $\mathbb{R}$ with $0 \in J$, and $f \in C^0(I \times J)$. We define $F : I \times \Upsilon \to L^2_{2\pi}$ (where $\Upsilon = \{w \in W^{1,2}_{2\pi} | \mathcal{R}(w) \subset J\}$) by

$$F(\lambda, w) = f(\lambda, w)(1 + \mathcal{C}w') + \mathcal{C}(f(\lambda, w)w') - f(\lambda, 0). \tag{3.1}$$

Equation (1.4) then becomes $F(\lambda, w) = 0$.

In Section 3.1 we give sufficient conditions on f for F to be in $C^k(I \times \Upsilon, L^2_{2\pi})$ and for F to be real-analytic; and also give a sufficient condition for $\partial^{0,1} F(\lambda, 0)$ to be Fredholm with index 0. Both of these conditions will be used in the following sections. We also observe that a necessary condition for $(\lambda^*, 0)$ to be a bifurcation point of (1.4) is that the function $q : I \to \mathbb{R}$ given by

$$q(\lambda) = -\frac{f^{(0,1)}(\lambda, 0)}{2f(\lambda, 0)} \tag{3.2}$$

takes a value in $\mathbb{N} \cup \{0\}$ at λ^*.

In Section 3.2 we apply Theorem 1.21 to (1.4) and find that a sufficient condition for $(\lambda^*, 0)$ to be a bifurcation point is that $f \in C^1(I \times J)$, $q(\lambda^*) \in \mathbb{N} \cup \{0\}$ and $(\lambda - \lambda^*)(q(\lambda) - q(\lambda^*))$ has the same sign on $I \setminus \{\lambda^*\}$. This hypothesis is very weak, but the conclusion (only that $(\lambda^*, 0)$ is a bifurcation point) is weak also.

In Section 3.3 we apply Theorem 1.22 to (1.4) and find that under the stronger hypotheses $f \in C^k$ $(k \geq 2)$, $q(\lambda^*) \in \mathbb{N} \cup \{0\}$ and $q'(\lambda^*) \neq 0$, a C^{k-1}-curve of solutions bifurcates from $(\lambda^*, 0)$. We are also able to give information about the behaviour of this curve in a neighbourhood of $(\lambda^*, 0)$.

Finally, in Section 3.4 we find, by applying the Morse lemma (Lemma 1.18) to the bifurcation equation (1.25), a sufficient condition for $(\lambda^*, 0)$ to be a bifurcation point of (1.4) where q has a strict local turning point at λ^*. In this case we find that two distinct C^1 branches of nontrivial solutions of (1.4) bifurcate from $(\lambda^*, 0)$. In so-doing, we show that the sufficient condition found in Section 3.2 is not a necessary condition.

3.1 Preliminaries

We address first the Fréchet differentiability of F.

Lemma 3.1. *Let I and J be open intervals in $\mathbb{R}$ with $0 \in J$. If $f \in C^k(I \times J)$ $(k \in \mathbb{N})$ then $F \in C^k(I \times \Upsilon, L^2_{2\pi})$. If f is real-analytic from $I \times J$ to $\mathbb{R}$ then F is real-analytic from $I \times \Upsilon$ to $L^2_{2\pi}$.*

Proof. By Theorems C.1 and C.2, if $f \in C^k(I \times J, \mathbb{R})$ then $F \in C^k(I \times \Upsilon, L^2_{2\pi})$. Hence we need only show that if f is real-analytic, then so too is F.

We show first that the map $(\lambda, w) \mapsto f(\lambda, w)$ is real-analytic from $I \times \Upsilon$ to $C^0_{2\pi}$. Let $\lambda_0 \in I$ and $w_0 \in \Upsilon$, and let $\delta = \min\{\operatorname{dist}(\lambda, \partial I), \operatorname{dist}(\mathcal{R}(w), \partial J)\}$ and $M = \{a + b \in \mathbb{R}^2 | a \in \{\lambda_0\} \times \mathcal{R}(w_0), |b| \leq \delta/4\}$. By (1.23) there exist $C > 1$ and $R \in (0,1)$ such that for all $n \in \mathbb{N}$ and all $(\lambda, y) \in M$,

$$\|\mathrm{d}^n f(\lambda, y)\|_{\mathcal{L}^n(\mathbb{R}^2,\mathbb{R})} \leq \frac{Cn!}{R^n}.$$

Let $(\lambda, w) \in \mathbb{R} \times W^{1,2}_{2\pi}$ with $\|(\lambda - \lambda_0, w - w_0)\|_{\mathbb{R} \times W^{1,2}_{2\pi}} \leq \delta/4$. Then $\lambda \in I$ since $|\lambda - \lambda_0| \leq \delta/4$ and $w \in \Upsilon$ since $\|w - w_0\|_\infty \leq \sqrt{2\pi}\|w - w_0\|_{W^{1,2}_{2\pi}} < \delta$. Let also

$t \in S^1$, $N \in \mathbb{N}$ and

$$f_N(\lambda, w(t)) = \sum_{n=0}^{N} \frac{1}{n!} \mathrm{d}^n f(\lambda_0, w_0(t))[(\lambda - \lambda_0, w(t) - w_0(t)), \ldots, (\lambda - \lambda_0, w(t) - w_0(t))].$$

By Taylor's theorem (1.19),

$$\begin{aligned} |f(\lambda, w(t)) - f_N(\lambda, w(t))| &\leq \frac{|(\lambda - \lambda_0, w(t) - w_0(t))|^{N+1}}{(N+1)!} \\ &\times \sup_{s \in [0,1]} \|\mathrm{d}^{N+1} f((1-s)(\lambda_0, w_0(t)) + s(\lambda, w(t)))\|_{\mathcal{L}^{N+1}(\mathbb{R}^2, \mathbb{R})}. \end{aligned}$$

Now for $s \in [0,1]$ and $t \in S^1$ we have that $((1-s)\lambda_0 + s\lambda, (1-s)w_0(t) + sw(t)) \in M$, so if $\|(\lambda_0 - \lambda, w_0 - w)\|_{\mathbb{R} \times W^{1,2}_{2\pi}} < \min\{R, \delta/4\}$ then

$$\begin{aligned} |f(\lambda, w(t)) - f_N(\lambda, w(t))| &\leq \frac{C(N+1)!|(\lambda - \lambda_0, w(t) - w_0(t))|^{N+1}}{R^{N+1}(N+1)!} \\ &\leq \frac{C(|\lambda - \lambda_0| + \|w - w_0\|_\infty)^{N+1}}{R^{N+1}} \to 0 \end{aligned}$$

so $(f_N(\lambda, w)) \to f(\lambda, w)$ in $C^0_{2\pi}$.

If we define a family of symmetric n-linear forms $m_n \in \mathcal{L}^n(\mathbb{R} \times W^{1,2}_{2\pi}, C^0_{2\pi})$ by

$$m_n((\lambda_1, w_1), \ldots, (\lambda_n, w_n))(t) = \frac{1}{n!} \mathrm{d}^n f(\lambda_0, w_0(t))[(\lambda_1, w_1(t)), \ldots, (\lambda_n, w_n(t))]$$

(recall that $\mathrm{d}^n f(\lambda_0, w_0(t)) \in \mathcal{L}^n(\mathbb{R}^2, \mathbb{R})$), then for $t \in S^1$

$$\begin{aligned} &m_n((\lambda - \lambda_0, w - w_0), \ldots, (\lambda - \lambda_0, w - w_0))(t) \\ &\quad = \frac{1}{n!} \mathrm{d}^n f(\lambda_0, w_0(t))[(\lambda - \lambda_0, w(t) - w_0(t)), \ldots, (\lambda - \lambda_0, w(t) - w_0(t))]; \end{aligned}$$

so by (1.21), $(\lambda, w) \mapsto f(\lambda, w)$ is real-analytic from $I \times \Upsilon$ to $C^0_{2\pi}$, as required.

By a similar (and simpler) argument, the map $(\lambda, w) \mapsto f(\lambda, 0)$ is real-analytic from $I \times \Upsilon$ to $C^0_{2\pi}$.

Now the operator $P : C^0_{2\pi} \times L^2_{2\pi} \to L^2_{2\pi}$ given by $P(u, v) = uv$ is real-analytic, since it equals its [three-term] Taylor series about each point. Since the maps $w \mapsto \mathcal{C}w'$ and $w \mapsto w'$ are real-analytic from $W^{1,2}_{2\pi}$ to $L^2_{2\pi}$ (being linear) it follows that the maps $(\lambda, w) \mapsto f(\lambda, w)(1+\mathcal{C}w')$ and $(\lambda, w) \mapsto f(\lambda, w)w'$ are compositions of real-analytic maps, hence are real-analytic from $I \times \Upsilon$ to $L^2_{2\pi}$ (see Section 1.5). Finally since $\mathcal{C}$ is linear, the map $w \mapsto \mathcal{C}(f(\lambda, w)w')$ is real-analytic from $W^{1,2}_{2\pi}$ to $L^2_{2\pi}$. The result follows. □

From now on we restrict our attention to even solutions of (1.4), in order to apply Theorem 1.21 and Theorem 1.22. Let

$$Z_0 = \{w \in L^2_{2\pi} | w \text{ is even}\} \quad Z_1 = \{w \in W^{1,2}_{2\pi} | w \text{ is even}\},$$

have the subset norms of $W^{1,2}_{2\pi}$ and $L^2_{2\pi}$ respectively. We consider F as a map from $I \times (\Upsilon \cap Z_1)$ to Z_0.

Lemma 3.2. *Suppose $f \in C^1(J)$ is such that $f(0) \neq 0$. If $F : \Upsilon \cap Z_1 \to Z_0$ is given by*

$$F(w) = f(w)(1 + \mathcal{C}w') + \mathcal{C}(f(w)w') - f(0) \tag{3.3}$$

then $\mathrm{d}F(0)$ is Fredholm with index 0.

Proof. By Lemma 3.1, $F \in C^1(\Upsilon \cap Z_1, Z_0)$, and by Corollary C.3,

$$\mathrm{d}F(0)h = hf'(0) + 2f(0)\mathcal{C}h'. \tag{3.4}$$

By the Theorem of M. Riesz (Theorem 1.7), $\mathrm{d}F(0)$ is a bounded linear operator.

Suppose first that $-f'(0)/2f(0) \notin \mathbb{N} \cup \{0\}$. Then $\mathrm{d}F(0)h = 0$ if and only if $h = 0$ [note that the set of eigenvalues of the map $h \mapsto \mathcal{C}h'$ is $\mathbb{N} \cup \{0\}$], so $\mathrm{d}F(0)$ is injective. Also, for $n \in \mathbb{N} \cup \{0\}$,

$$\mathrm{d}F(0)\left[\frac{\Phi_n}{f'(0) + 2nf(0)}\right] = \Phi_n, \tag{3.5}$$

where $\Phi_n(x) = \cos nx$, so $\mathrm{d}F(0)$ maps onto a total subset of Z_0 [a set with span dense in Z_0], and it follows from an argument involving Parseval's equation and the comparison test that $\mathrm{d}F(0)$ is surjective. The result follows

Now suppose that $-f'(0)/2f(0) = k \in \mathbb{N} \cup \{0\}$. Then $\mathrm{d}F(0)h = 0$ if and only if $h \in \operatorname{span}\{\Phi_k\}$, so $\mathcal{N}(\mathrm{d}F(0)) = \operatorname{span}\{\Phi_k\}$. Also, (3.5) holds for all $n \in \mathbb{N} \cup \{0\} \setminus \{k\}$, but for all $w \in Z_1$, $\mathrm{d}F(0)w \neq \Phi_k$. Hence $\mathrm{d}F(0)$ maps onto a total subset of $\{w \in Z_0 |\, \langle w, \Phi_k \rangle_2 = 0\}$. Hence by an argument involving Parseval's equation and the comparison test, $\mathcal{R}(\mathrm{d}F(0)) = \{w \in Z_0 |\, \langle w, \Phi_k \rangle_2 = 0\}$.

The result follows. □

Finally in this section we give a necessary condition for $(\lambda^*, 0)$ to be a bifurcation point of (1.4).

Lemma 3.3. *Let I and J be open intervals in $\mathbb{R}$ with $0 \in J$, and $f \in C^1(I \times J)$. If $f(\lambda^*, 0) \neq 0$ and $q^* = q(\lambda^*) \notin \mathbb{N} \cup \{0\}$ (where q is as given in (3.2)) then $(\lambda^*, 0)$ is not a bifurcation point of (1.4).*

Proof. By the proof of Lemma 3.2, $\partial^{0,1}F(\lambda^*, 0)$ is a bijective, bounded linear operator. Hence by the bounded inverse theorem, $\partial^{0,1}F(\lambda^*, 0)$ is a homeomorphism and $(\lambda^*, 0)$ is not a bifurcation point of (1.4) by Lemma 1.20. □

3.2 Odd eigenvalue crossing number bifurcation

The following theorem shows, using the concept of eigenvalue crossing numbers defined in Section 1.6, that a large class of points $(\lambda, 0)$ not excluded by Lemma 3.3 are bifurcation points of (1.4). As in the previous section, we restrict our attention to finding even solutions of (1.4): this ensures that all the eigenvalues of $\partial^{0,1}F(\lambda^*, 0)$ are simple.

Theorem 3.4. *Let $\lambda^* \in \mathbb{R}$ and let $I \times J$ be a neighbourhood of $(\lambda^*, 0)$ in $\mathbb{R}^2$. Suppose $f \in C^1(I \times J)$ is bounded away from 0. If $q^* \in \mathbb{N} \cup \{0\}$ and one of the* *following holds:*

$$(\forall\lambda \in I \setminus \{\lambda^*\})(\lambda - \lambda^*)(q(\lambda) - q^*) \;>\; 0 \tag{3.6}$$

$$(\forall\lambda \in I \setminus \{\lambda^*\})(\lambda - \lambda^*)(q(\lambda) - q^*) \;<\; 0 \tag{3.7}$$

then $(\lambda^*, 0)$ *is a bifurcation point of (1.4).*

Proof. By Corollary C.3,

$$\partial^{0,1}F(\lambda^*, 0)h = hf^{(0,1)}(\lambda^*, 0) + 2f(\lambda^*, 0)\mathcal{C}h'. \tag{3.8}$$

By Theorem 3.1, $F \in C^1(I \times \Upsilon, L^2_{2\pi})$ because $f \in C^1(I \times J)$. By Lemma 3.2, $\partial^{0,1}F(\lambda, 0)$ is Fredholm with index 0 for all $\lambda \in I$.

We find, for $\lambda \in I$, the eigenvalues of $\partial^{0,1}F(\lambda, 0)$. By (3.8), μ is an eigenvalue of $\partial^{0,1}F(\lambda, 0)$ if and only if

$$-\frac{f^{(0,1)}(\lambda, 0) - \mu}{2f(\lambda, 0)} \in \mathbb{N} \cup \{0\}:$$

i.e. if and only if $\mu = \mu_n(\lambda) = f^{(0,1)}(\lambda, 0) + 2nf(\lambda, 0)$ for some $n \in \mathbb{N} \cup \{0\}$. In particular, $\dim(\mathcal{N}(\partial^{0,1}F(\lambda^*, 0))) = 1$ and 0 is an isolated [simple] eigenvalue of $\partial^{0,1}F(\lambda^*, 0)$. The eigenspace corresponding to $\mu_n(\lambda)$ is $\text{span}\{\Phi_n\}$.

Since $f \in C^1(I \times J)$, we have that as $\lambda \to \lambda^*$, $\mu_n(\lambda) \to f^{(0,1)}(\lambda^*, 0) + 2nf(\lambda^*, 0)$. It follows that the only eigenvalue of $\partial^{0,1}F(\lambda, 0)$ which converges to 0 as $\lambda \to \lambda^*$ is $\mu_{q^*}(\lambda) = 2f(\lambda, 0)(q^* - q(\lambda))$.

If $f > 0$ on $I \times J$ then $\mu_{q^*} < 0$ if and only if $q^* < q(\lambda)$, so the number of negative eigenvalues of $\partial^{0,1}F(\lambda, 0)$ which converge to 0 as $\lambda \to \lambda^*$ is given by

$$\sigma(\lambda) = \begin{cases} 1 & q^* < q(\lambda) \\ 0 & q^* > q(\lambda). \end{cases}$$

It follows that

$$\chi(\partial^{0,1}F(\lambda,0),\lambda^*) = \begin{cases} -1 & \text{if (3.6) holds} \\ 1 & \text{if (3.7) holds;} \end{cases}$$

and similarly $\chi(\partial^{0,1}F(\lambda,0),\lambda^*) = \pm 1$ if $f < 0$ on $I \times J$. In all cases the crossing number is odd; so by Theorem 1.21, $(\lambda^*, 0)$ is a bifurcation point of (1.4), as required. □

Remarks 3.5. (a) If $q^* \in \mathbb{N} \cup \{0\}$, it follows that $f(\lambda^*, 0) \neq 0$, so by the continuity of f there exists a neighbourhood $I \times J$ of $(\lambda^*, 0)$ such that f is bounded away from 0 on $I \times J$. The same remark holds for Theorems 3.7 and 3.13 also.

(b) The conditions given in (3.6-3.7) each ensure that $q(\lambda) - q^*$ changes sign as λ passes λ^*.

■

Example 3.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = 1 - y(y+2)(1+\lambda^3).$$

Then $f(\lambda, 0) = 1$ for all λ. Also we have that $f^{(0,1)}(\lambda, y) = -2(1+y)(1+\lambda^3)$, so that $f^{(0,1)}(\lambda, 0) = -2(1+\lambda^3)$ and $q(\lambda) = 1 + \lambda^3$. Hence $q(0) = 1$ and for all $\lambda \neq 0$,

$$(q(\lambda) - q(0))(\lambda - 0) = \lambda^4 > 0;$$

so $(0, 0)$ is a bifurcation point of (1.4). ■

3.3 Crandall-Rabinowitz transversality

If we have better regularity information about f and a stronger condition on the behaviour of q then we are able to say more about the solution set of (1.4) near

$(\lambda^*, 0)$: namely

Theorem 3.7. *Let $\lambda^* \in \mathbb{R}$ and let $I \times J$ be a neighbourhood of $(\lambda^*, 0)$ in $\mathbb{R}^2$. Suppose $f \in C^k(I \times J)$ where $k \geq 2$. If $q^* \in \mathbb{N} \cup \{0\}$ and $q'(\lambda^*) \neq 0$ then a unique C^{k-1}-curve $\mathfrak{B}$ of nontrivial even solutions of (1.4) bifurcates from the line $I \times \{0\} \subset \mathbb{R} \times W^{1,2}_{2\pi}$ of trivial solutions at $(\lambda^*, 0)$.*

If $q^* \in \mathbb{N}$ then all solutions on $\mathfrak{B}$ in a punctured neighbourhood of $(\lambda^*, 0)$ have fundamental period $2\pi/q^*$; if $q^* = 0$ then all solutions on $\mathfrak{B}$ in a neighbourhood of $(\lambda^*, 0)$ are constant functions.

If f is real-analytic then so is $\mathfrak{B}$.

Proof. We show that $\partial^{0,1}F(\lambda^*, 0)$ satisfies the hypotheses of Theorem 1.22.

Note first that by Lemma 3.1, $F \in C^k(I \times \Upsilon, L^2_{2\pi})$ because $f \in C^k(I \times J)$; and that if f is real-analytic, then so too is F.

By the proof of Lemma 3.2, we have that $\mathcal{N}(\partial^{0,1}F(\lambda^*, 0)) = \operatorname{span}\{\Phi_{q^*}\}$ and $\mathcal{R}(\partial^{0,1}F(\lambda^*, 0)) = \{w \in Z_0 \,|\, \langle w, \Phi_{q^*}\rangle_2 = 0\}$, so $\partial^{0,1}F(\lambda^*, 0)$ is Fredholm with index 0, and has a one-dimensional null space.

Now for all λ for which $f(\lambda, 0) \neq 0$,

$$q'(\lambda) = \frac{f^{(1,0)}(\lambda, 0) f^{(0,1)}(\lambda, 0) - f(\lambda, 0) f^{(1,1)}(\lambda, 0)}{2 f(\lambda, 0)^2}, \tag{3.9}$$

so by (C.7),

$$\begin{aligned}&\partial^{1,1}F(\lambda^*, 0)[\Phi_{q^*}, 1] \\ &= \left[f^{(1,1)}(\lambda^*, 0) - \frac{f^{(0,1)}(\lambda^*, 0)}{f(\lambda^*, 0)} f^{(1,0)}(\lambda^*, 0) \right] \Phi_{q^*} \notin \mathcal{R}(\partial^{0,1}F(\lambda^*, 0)),\end{aligned} \tag{3.10}$$

since $q'(\lambda^*) \neq 0$. Hence $\partial^{0,1}F(\lambda^*, 0)$ satisfies the hypotheses of Theorem 1.22, and we conclude that a unique curve $\mathfrak{B}$ of even solutions of (1.4) bifurcates from the line of trivial solutions at $(\lambda^*, 0)$, as required. The regularity of $\mathfrak{B}$ follows from the same theorem and from Lemma 3.1

$\mathfrak{B}$ is given by (1.28) as $\{\mathcal{B}(t) | t \in (-\varepsilon, \varepsilon)\}$ (for some $\varepsilon > 0$), where $\mathcal{B} \in C^{k-1}((-\varepsilon, \varepsilon), I \times (\Upsilon \cap Z_1))$ ($\mathcal{B}$ is real-analytic if f is) is given by

$$\mathcal{B}(t) = (\lambda^* + \mu(t), t\Phi_{q^*} + \gamma(\lambda^* + \mu(t), t\Phi_{q^*})). \tag{3.11}$$

In order to prove the periodicity claim for solutions in $\mathfrak{B}$, we define Z_{0,q^*} and Z_{1,q^*} as follows: if $q^* \in \mathbb{N}$ then

$$\begin{aligned} Z_{0,q^*} &= \left\{ w \in Z_0 \,\middle|\, w \text{ is } \frac{2\pi}{q^*}\text{-periodic} \right\} \\ Z_{1,q^*} &= \left\{ w \in Z_1 \,\middle|\, w \text{ is } \frac{2\pi}{q^*}\text{-periodic} \right\}, \end{aligned}$$

with the subspace norms of $W^{1,2}_{2\pi}$ and $L^2_{2\pi}$ respectively. Otherwise Z_{0,q^*} and Z_{1,q^*} are the set of constant functions on $\mathbb{R}$.

We regard F as a map from $I \times (\Upsilon \cap Z_{1,q^*})$ to Z_{0,q^*} and find as before that $\partial^{0,1}F(\lambda^*, 0)$ is a Fredholm operator with index 0 and has null space $\operatorname{span}\{\Phi_{q^*}\}$; and that the transversality condition (3.10) holds. Hence a unique C^{k-1} (or real-analytic) curve of solutions $(\lambda, w) \in \mathbb{R} \times Z_{1,q^*}$ bifurcates from the line of trivial solutions at $(\lambda^*, 0)$. Since the bifurcating curve $\mathfrak{B}$ found above is unique, it must therefore, in a neighbourhood of $(\lambda^*, 0)$, lie in $\mathbb{R} \times Z_{1,q^*}$.

If $q^* \neq 0$ then (3.11) shows that in a punctured neighbourhood of $(\lambda^*, 0)$, all solutions on $\mathfrak{B}$ are nonconstant, since γ maps into $\mathcal{R}(\partial^{0,1}F(\lambda^*, 0)) = \{u \in Z_{0,q^*} \,|\, \langle u, \Phi_{q^*}\rangle_2 = 0\}$. Hence in a punctured neighbourhood of $(\lambda^*, 0)$, all solutions on $\mathfrak{B}$ have fundamental period $2\pi/q^*$. □

In the case where f and λ^* satisfy Theorem 3.7, we shall say that the bifurcation and the bifurcation point $(\lambda^*, 0)$ are of **Crandall-Rabinowitz type**.

The bifurcation equation (1.26) is given by

$$\beta(\mu, t) = \langle \Phi_{q^*}, F(\lambda^* + \mu, t\Phi_{q^*} + \gamma(\lambda^* + \mu, t\Phi_{q^*}))\rangle_2 = 0 \tag{3.12}$$

in a neighbourhood of $(0,0)$ in $\mathbb{R}^2$.

We next examine the behaviour of $\mathfrak{B}$ near $(\lambda^*, 0)$. The cases $q^* = 0$ and $q^* \neq 0$ are treated separately.

Theorem 3.8. *Suppose f and λ^* satisfy Theorem 3.7 and $q^* \neq 0$. Then $\mathfrak{B}$, given by (3.11), is not transcritical.*

Proof. Suppose (λ, w) is a nontrivial solution on $\mathfrak{B}$. By (3.11), $w = t\Phi_{q^*} + \gamma(\lambda^* + \mu(t), t\Phi_{q^*})$ for some $t \neq 0$, so w cannot be π/q^*-periodic as its Fourier series contains a nonzero multiple of Φ_{q^*} (recall that γ maps into $\mathcal{R}(\partial^{0,1}F(\lambda^*,0)) = \{u \in Z_0 | \langle u, \Phi_{q^*}\rangle_2 = 0\}$).

Now the function $w_{q^*} \in W^{1,2}_{2\pi}$ given by $w_{q^*}(x) = w(x - \pi/q^*)$ is even, since for $x \in \mathbb{R}$, $w_{q^*}(-x) = w(-x-\pi/q^*) = w(x+\pi/q^*) = w(x-\pi/q^*) = w_{q^*}(x)$. Also, as with $\mathcal{F}$, $F(\lambda, \cdot)$ is translation equivariant; so for all x

$$F(\lambda, w_{q^*})(x) = 0 = F(\lambda, w)\left(x - \frac{\pi}{q^*}\right).$$

Hence if (λ, w) is a nontrivial even solution of (1.4) then (λ, w_{q^*}) is a distinct, nontrivial even solution. Hence $\mathfrak{B}$ cannot be transcritical. $\square$

In fact Theorem 1.24 gives sufficient conditions for $\mathfrak{B}$ to be supercritical or subcritical.

Theorem 3.9. *Suppose $f \in C^3(I \times J)$ and λ^* satisfy Theorem 3.8. If $B(\lambda^*)/(q'(\lambda^*)f(\lambda^*,0)) > 0$ where*

$$B(\lambda) = \frac{1}{4}f^{(0,3)}(\lambda,0) + \frac{f^{(0,2)}(\lambda,0)^2}{4f^{(0,1)}(\lambda,0)} - \frac{f^{(0,2)}(\lambda,0)f^{(0,1)}(\lambda,0)}{2f(\lambda,0)} + \frac{f^{(0,1)}(\lambda,0)^3}{2f(\lambda,0)^2}, \quad (3.13)$$

then $\mathfrak{B}$ is supercritical. If $B(\lambda^*)/(q'(\lambda^*)f(\lambda^*,0)) < 0$ then $\mathfrak{B}$ is subcritical.

Proof. Since by Theorem 3.8, $\mu'(0) = 0$, we have by Theorem 1.24 that

$$\mu''(0) = -\frac{\beta^{(0,3)}(0,0)}{3\beta^{(1,1)}(0,0)}, \quad (3.14)$$

where β is given by (3.12). The calculations in Appendix C show that $\beta^{(0,3)}(0,0) = 3\pi B(\lambda^*)$ and

$$\beta^{(1,1)}(0,0) = \pi\left(f^{(1,1)}(\lambda^*,0) - \frac{f^{(1,0)}(\lambda^*,0)f^{(0,1)}(\lambda^*,0)}{f(\lambda^*,0)}\right) = -2\pi q'(\lambda^*)f(\lambda^*,0),$$

so that

$$\mu''(0) = \frac{B(\lambda^*)}{2q'(\lambda^*)f(\lambda^*,0)}$$

[note that $\beta^{(1,1)}(0,0) \neq 0$ by hypothesis (see (3.9))]. The result now follows from the remarks in Section 1.6. $\square$

Example 3.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = 1 - y(2+y^2)(3-2\lambda^2).$$

Then $f(\lambda,0) = 1$ for all λ. We also have that for all λ and y, $f^{(0,1)}(\lambda,y) = -(2+3y^2)(3-2\lambda^2)$, so that $f^{(0,1)}(\lambda,0) = -2(3-2\lambda^2)$ and $q(\lambda) = 3-2\lambda^2$. We show that (1.4) has bifurcation points of Crandall-Rabinowitz type at ± 1. We have $q(\pm 1) = 1$ and for all λ, $q'(\lambda) = -4\lambda$, so $q'(\pm 1) = \mp 4$. Hence by Theorem 3.7, real-analytic curves of even nonconstant solutions of (1.4) with fundamental period 2π bifurcate from $(\pm 1, 0)$.

Now for all λ and y, $f^{(0,2)}(\lambda,y) = -6y(3-2\lambda^2)$ and $f^{(0,3)}(\lambda,y) = -6(3-2\lambda^2)$. It follows that $f^{(0,1)}(\pm 1,0) = -2$, $f^{(0,2)}(\pm 1, 0) = 0$ and $f^{(0,3)}(\pm 1,0) = -6$; so $B(\pm 1) = -11/2$. Hence

$$\frac{B(1)}{q'(1)f(1,0)} = \frac{-11/2}{-4} > 0,$$

so by Theorem 3.9, the curve bifurcating from $(1,0)$ is supercritical; and similarly the curve bifurcating from $(-1,0)$ is subcritical. $\blacksquare$

In the case where $B(\lambda^*) = 0$, further calculation (of the fifth order Taylor polynomial of β) and correspondingly stronger regularity conditions on f will give sufficient conditions for sub- and super- criticality of $\mathfrak{B}$.

If $q^* = 0$, transcritical bifurcating curves are possible.

Theorem 3.11. *Suppose that $f \in C^k(I \times J)$ $(k \geq 2)$ and λ^* satisfy Theorem 3.7 and are such that $q^* = 0$. Suppose that for all $n \in \{1, \ldots, k-1\}$ $f^{(0,n)}(\lambda^*, 0) = 0$, and $f^{(0,k)}(\lambda^*, 0) \neq 0$. If k is even then $\mathfrak{B}$ is transcritical. If k is odd then $\mathfrak{B}$ is supercritical if $f^{(0,k)}(\lambda^*, 0)/f^{(1,1)}(\lambda^*, 0) < 0$ and subcritical if $f^{(0,k)}(\lambda^*, 0)/f^{(1,1)}(\lambda^*, 0) > 0$.*

Proof. If $q^* = 0$ then, by Theorem 3.7 the solutions on $\mathfrak{B}$ in a neighbourhood of $(\lambda^*, 0)$ are constant functions. We shall denote such functions by the values they take. We have

$$F(\lambda, c) = f(\lambda, c) - f(\lambda, 0). \tag{3.15}$$

Considering F as a map from the space of constant functions into itself, we find that the map γ given in (3.11) has range $\{0\}$, so $\beta(\mu, t) = \langle \Phi_0, F(\lambda^* + \mu, t\Phi_0) \rangle_2 = 2\pi\{f(\lambda^* + \mu, t) - f(\lambda^* + \mu, 0)\}$. It follows that for all $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ with $m + n \leq k$,

$$\beta^{(m,n)}(0,0) = 2\pi f^{(m,n)}(\lambda^*, 0),$$

Hence by Theorem 1.24, $\mu^{(n)}(0) = 0$ for all $n \in \{0, \ldots, k-2\}$ and

$$\mu^{(k-1)}(0) = -\frac{\beta^{(0,k)}(0,0)}{k\beta^{(1,1)}(0,0)} = -\frac{f^{(0,k)}(\lambda^*, 0)}{kf^{(1,1)}(\lambda^*, 0)}.$$

The theorem follows from the remarks above Theorem 1.24. $\square$

Example 3.12. As seen in (3.15), the constant solutions (λ, c) of (1.4) are those for which $f(\lambda, c) = f(\lambda, 0)$. It is thus easy to provide examples of supercritical, subcritical and transcritical bifurcations of Crandall-Rabinowitz type.

1. If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(\lambda, y) = 1 + y(y - \lambda)$$

then $f(\lambda, 0) = 1$ and $f^{(0,1)}(\lambda, 0) = -\lambda$ for all λ. Hence (1.4) has a bifurcation point of constant solutions at $(0, 0)$. We also have $f^{(0,2)}(0, 0) = 2$, so the bifurcation is transcritical. [The set of constant solutions of (1.4) is $\{(\lambda, 0), (\lambda, \lambda) | \lambda \in \mathbb{R}\}$.]

2. If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(\lambda, y) = y^3 + (1 - \lambda^2)y + 1$$

then for all λ, $f(\lambda, 0) = 1$ and $f^{(0,1)}(\lambda, 0) = 1 - \lambda^2$. Hence (1.4) has bifurcation points of constant solutions at $(-1, 0)$ and at $(1, 0)$. We also have that $f^{(0,2)}(\lambda, 0) = 0$, $f^{(1,1)}(\lambda, 0) = -2\lambda$ and $f^{(0,3)}(\lambda, 0) = 6$. Hence the bifurcation at $(-1, 0)$ is subcritical, whilst the one at $(1, 0)$ is supercritical. [The set of constant solutions of (1.4) is $\{(\lambda, 0) | \lambda \in \mathbb{R}\} \cup \{(\lambda, \pm\sqrt{\lambda^2 - 1}) | \lambda \in \mathbb{R} \setminus (-1, 1)\}$.]

■

3.4 Crossing number 0: double bifurcation

Theorem 3.4 shows that a sufficient condition for $(\lambda^*, 0)$ to be a bifurcation point is that the product $(q(\lambda) - q^*)(\lambda - \lambda^*)$ has one sign in a punctured neighbourhood of λ^*. The following theorem shows that this condition is not necessary. We investigate a particular case where $f \in C^4(I \times J)$ and λ^* satisfy $q^* \in \mathbb{N}$ and $q'(\lambda^*) = 0$ by calculating the Taylor series of the bifurcation equation (3.12).

Theorem 3.13. *Suppose $f \in C^k(I \times J)$, where $k \geq 4$. If $q^* \in \mathbb{N}$, $q'(\lambda^*) = 0$ and $f(\lambda^*, 0)q''(\lambda^*)/B(\lambda^*) > 0$, where B is given by (3.13), then two distinct C^{k-3} curves, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, of nonconstant even solutions of (1.4) bifurcate from the line of trivial solutions at $(\lambda^*, 0)$.*

All solutions on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in a punctured neighbourhood of $(\lambda^*, 0)$ have fundamental period $2\pi/q^*$.

If f is real-analytic, then so too are $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

Proof. If f and λ^* satisfy the hypotheses of the theorem then by (3.9),

$$\begin{aligned} q''(\lambda^*) &= \frac{f(\lambda^*,0)\{f^{(2,0)}(\lambda^*,0)f^{(0,1)}(\lambda^*,0) - f^{(2,1)}(\lambda^*,0)f(\lambda^*,0)\}}{2f(\lambda^*,0)^3} \\ &\quad - \frac{f^{(1,0)}(\lambda^*,0)\{f^{(1,0)}(\lambda^*,0)f^{(0,1)}(\lambda^*,0) - f^{(1,1)}(\lambda^*,0)f(\lambda^*,0)\}}{2f(\lambda^*,0)^3} \\ &= \frac{f^{(2,0)}(\lambda^*,0)f^{(0,1)}(\lambda^*,0) - f^{(2,1)}(\lambda^*,0)f(\lambda^*,0)}{2f(\lambda^*,0)^2}. \end{aligned}$$

A long calculation (see Appendix C) shows that if $q^* \in \mathbb{N}$ and $q'(\lambda^*) = 0$, then the C^k (or real-analytic) function β, given by (3.12), has Taylor series

$$\beta(\mu,t) = \frac{\pi t}{2}(-2f(\lambda^*,0)q''(\lambda^*)\mu^2 + B(\lambda^*)t^2 + \rho') \tag{3.16}$$

about $(0,0)$, where B is given by (3.13) and $\rho' = O(|\mu^3| + |\mu^2 t| + |\mu t^2| + |t^3|)$ (see the remarks following (C.10)). It follows that the C^{k-1} (or real-analytic) function h given by (1.27) has Taylor series $h(\mu,t) = \frac{\pi}{2}(-f(\lambda^*,0)q''(\lambda^*)\mu^2 + B(\lambda^*)t^2 + \rho')$ about $(0,0)$.

Provided the coefficients of μ^2 and t^2 are nonzero, $(0,0)$ is a nondegenerate critical point (see Section 1.5) of h. Clearly this holds if $f(\lambda^*,0)q''(\lambda^*)/B(\lambda^*) > 0$.

Hence, if f and λ^* satisfy the hypotheses of the theorem, then by the Morse lemma (Lemma 1.18), there exists a local C^{k-3} (or real-analytic) diffeomorphism Γ on a neighbourhood of $(0,0)$ such that $h(\Gamma(\nu,\tau)) = \frac{\pi}{2}(-2f(\lambda^*,0)q''(\lambda^*)\nu^2 + B(\lambda^*)\tau^2)$. It follows that $h(\Gamma(\nu,\tau)) = 0$ if and only if

$$\nu = \pm\tau\sqrt{\frac{2f(\lambda^*,0)q''(\lambda^*)}{B(\lambda^*)}}; \tag{3.17}$$

so $h(\mu, t) = 0$ (and equivalently, for $t \neq 0$, $\beta(\mu, t) = 0$) if and only if

$$(\mu, t) = \Gamma\left(\pm\tau\sqrt{\frac{2f(\lambda^*, 0)q''(\lambda^*)}{B(\lambda^*)}}, \tau\right) : \qquad (3.18)$$

i.e. if and only if (μ, t) lies on one of two C^{k-3} (or real-analytic) curves.

Since $f(\lambda^*, 0)q''(\lambda^*)/B(\lambda^*) \neq 0$, it follows from (3.16) that the curves found in (3.18) must, at $(0, 0)$, be tangent to the (distinct) lines given in (3.17); so the curves must be distinct.

The curves given in (3.18) define, on a neighbourhood $(-\varepsilon, \varepsilon)$, functions μ_1 and μ_2 of t. The Lyapunov-Schmidt reduction (Lemma 1.23) then shows that the nontrivial solutions of (1.4) are given, in a neighbourhood of $(\lambda^*, 0)$ as $\{\mathcal{B}_1(t), \mathcal{B}_2(t) | t \in (-\varepsilon, \varepsilon) \setminus \{0\}\}$, where

$$\begin{aligned} \mathcal{B}_1(t) &= (\lambda^* + \mu_1(t), t\Phi_{q^*} + \gamma(\lambda^* + \mu_1(t), t\Phi_{q^*})) \\ \mathcal{B}_2(t) &= (\lambda^* + \mu_2(t), t\Phi_{q^*} + \gamma(\lambda^* + \mu_2(t), t\Phi_{q^*})) : \end{aligned}$$

i.e. the nontrivial solutions of (1.4) are given, in a neighbourhood of $(\lambda^*, 0)$, by two transcritical C^{k-3} (or real-analytic) curves, $\mathfrak{B}_1$ and $\mathfrak{B}_2$, as required.

Remarks similar to those at the end of the proof of Theorem 3.7 show that all solutions on $\mathfrak{B}_1$ and $\mathfrak{B}_2$ in a punctured neighbourhood of $(\lambda^*, 0)$ are nonconstant and have fundamental period $2\pi/q^*$. □

Remark 3.14. By the proof of Theorem 3.8, if $(\lambda, w) \in \mathfrak{B}_1$ then $(\lambda, w_{q^*}) \in \mathfrak{B}_2$. ■

In the case where f and λ^* satisfy Theorem 3.13, we shall call the bifurcation and the bifurcation point $(\lambda^*, 0)$ **double**.

Example 3.15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = (\lambda - y)^2 + \lambda - 2y + 1.$$

Then f is real-analytic, and for all λ and y, $f(\lambda, 0) = \lambda^2 + \lambda + 1$, $f^{(0,1)}(\lambda, y) = 2y - 2\lambda - 2$ and $f^{(0,1)}(\lambda, 0) = -2 - 2\lambda$. It follows that

$$q(\lambda) = \frac{\lambda + 1}{\lambda^2 + \lambda + 1},$$

whence

$$q'(\lambda) = -\frac{\lambda(\lambda + 2)}{(\lambda^2 + \lambda + 1)^2}, \quad q''(\lambda) = \frac{2(\lambda^3 + 3\lambda^2 - 1)}{(\lambda^2 + \lambda + 1)^3}.$$

It follows that $q(0) = 1$, $q'(0) = 0$ and $q''(0) = -2$. We also have that $f(0, 0) = 1$ and for all λ and y, $f^{(0,2)}(\lambda, y) = 2$ and $f^{(0,3)}(\lambda, y) = 0$; and hence $B(0) = -5/2$.

Hence f and $\lambda^* = 0$ satisfy the hypotheses of Theorem 3.13, so two distinct real-analytic curves of nonconstant even solutions of (1.4) of fundamental period 2π bifurcate from the line of trivial solutions at $(0, 0)$. ■

Chapter 4

Global Bifurcation Theory

In this chapter we show, using the global analytic bifurcation theorem (Theorem 1.25), that if $(\lambda^*, 0)$ is a double bifurcation point or a bifurcation point of Crandall-Rabinowitz type of (1.4) and f is real-analytic and bounded away from 0 on a neighbourhood P of $(\lambda^*, 0)$ then a curve $\mathfrak{B}$ of solutions of (1.4) bifurcating from $(\lambda^*, 0)$ may be continued globally.

In Section 4.1 we give conditions for the hypotheses of Theorem 1.25 to hold, and then apply the theorem to (1.4).

Theorem 1.25 v gives three possibilities for the continuation of $\mathfrak{B}$. In Section 4.2, we give, using the Bernoulli condition (Theorem 2.19 (β)) and an observation about $\mathcal{F}$, conditions which prevent the first two of these; so ensuring that the continuation of $\mathfrak{B}$ forms a loop.

4.1 Application of the global analytic bifurcation theorem

We exhibit first sufficient conditions on f for hypothesis I of Theorem 1.25 to be satisfied. Lemma 4.1 - Theorem 4.4 give conditions for $\partial^{0,1}F(\lambda, 0)$ to be Fredholm with index 0.

Lemma 4.1. *Suppose $\phi \in L^\infty_{2\pi}$ is even and essentially bounded away from zero. Then the operator $\Psi : Z_1 \to Z_0$ given by $\Psi(h) = (Ch' + h)\phi$ is a linear homeo-*

morphism.

Proof. Let $h \in Z_1$. Then

$$\|\Psi(h)\|_2 \leq \|\phi\|_\infty \|Ch' + h\|_2 \leq \|\phi\|_\infty (\|h'\|_2 + \|h\|_2) \leq C\|h\|_{W^{1,2}_{2\pi}},$$

where C is a constant independent of h; so $\Psi : Z_1 \to Z_0$ is bounded. We show that it is a bijection.

Let $u \in Z_0$. Then for some coefficients a_n such that $\sum_{n=0}^\infty a_n^2 < \infty$, $u(x) = \sum_{n=0}^\infty a_n \cos nx$. If h is given by $h(x) = \sum_{n=0}^\infty \frac{1}{n+1} a_n \cos nx$ then $h \in Z_1$ since the weak derivative h' is given by $h'(x) = -\sum_{n=1}^\infty \frac{n}{n+1} a_n \sin nx$ almost everywhere and

$$\sum_{n=1}^\infty \frac{n}{n+1} a_n^2 < \sum_{n=1}^\infty a_n^2 < \infty.$$

Since then $h + Ch' = u$, it follows that the map $h \mapsto h + Ch'$ is surjective from Z_1 onto Z_0.

Now since ϕ is bounded and bounded away from zero, $u/\phi \in Z_0$ for all $u \in Z_0$. It follows from the surjectivity of $h \mapsto h+Ch'$ from Z_1 onto Z_0 that Ψ is surjective also.

Suppose, for some $h \in Z_1$ (with $h(x) = \sum_{n=0}^\infty c_n \cos nx$) that $\Psi(h) = 0$. Then for almost every x, $h(x) + Ch'(x) = 0$; and so

$$0 = \int_{-\pi}^{\pi} h(x)(h(x) + Ch'(x))\,\mathrm{d}x = 2\pi c_0^2 + \pi \sum_{n=1}^\infty (n+1) c_n^2$$

if and only if $c_n = 0$ for all n: i.e. if and only if $h = 0$. The injectivity of Ψ follows. $\square$

Lemma 4.2. *(generalisation of theorem 3.4 in [4]) Let J be an open interval in $\mathbb{R}$ and $f \in C^{1,1}_{\mathrm{loc}}(J)$. If $v \in \Upsilon \cap Z_1$ then $\mathcal{F}$ is sequentially continuous at v from $\Upsilon \cap Z_1$ with the weak topology into the space of even $L^p_{2\pi}$-functions ($1 \leq p < \infty$) with the strong topology.*

Proof. It will suffice to show, for each p, that if $(v_n) \rightharpoonup v$ in Z_1 then, for a subsequence, $(\mathcal{F}(v_{n_i})) \to \mathcal{F}(v)$ in $L^p_{2\pi}$. In this case, suppose that $(\mathcal{F}(v_n)) \not\to \mathcal{F}(v)$ in $L^p_{2\pi}$. Then there exists $\varepsilon > 0$ and a subsequence (v_{n_N}) of (v_n) such that for all n, $\|\mathcal{F}(v_{n_N}) - \mathcal{F}(v)\|_p \geq \varepsilon$. But then $(v_{n_N}) \rightharpoonup v$ in Z_1, so has a subsequence such that $(\mathcal{F}(v_{n_{N_i}})) \to \mathcal{F}(v)$ in $L^p_{2\pi}$: a contradiction.

Note first that if $w \in Z_1$, then $\mathcal{F}(w)$ is even. Let $(v_n) \rightharpoonup v$ in Z_1. Since $W^{1,2}_{2\pi}$ is compactly embedded in $L^\infty_{2\pi}$, a subsequence converges strongly to v in $L^\infty_{2\pi}$, and we may choose a further subsequence such that $\bigcup_{n\in\mathbb{N}} \mathcal{R}(v_n) \subset \hat{J}$, where $\hat{J}$ is a compact subinterval of J. Also, $(v_n') \rightharpoonup v'$ in $L^2_{2\pi}$. Theorem 3.4 in [4] shows that the operator $\mathcal{Q}$ given by

$$\mathcal{Q}(u)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(u(x)-u(y))u'(y)}{\tan\frac{x-y}{2}}\,dy.$$

is sequentially continuous from Z_1 with the weak topology into the space of even $L^p_{2\pi}$-functions $(1 \leq p < \infty)$ with the strong topology. It follows that for a further subsequence, $(\mathcal{Q}(v_n)(x)) \to \mathcal{Q}(v)(x)$ for almost every x. For such x,

$$\begin{aligned}
&\mathcal{F}(v_n)(x) - \mathcal{F}(v)(x)\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\{f(v_n(x)) - f(v_n(y))\}v_n'(y) - \{f(v(x)) - f(v(y))\}v'(y)}{\tan\frac{x-y}{2}}\,dy\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v_n'(y)\int_y^x \{f'(v_n(t)) - f'(v(t))\}\, v_n'(t)\,dt}{\tan\frac{x-y}{2}}\,dy\\
&\quad+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v_n'(y)\int_y^x f'(v(t))\,\{v_n'(t) - v'(t)\}\,dt}{\tan\frac{x-y}{2}}\,dy\\
&\quad+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\{v_n'(y) - v'(y)\}\{f(v(x)) - f(v(y))\}}{\tan\frac{x-y}{2}}\,dy\\
&= I_n(x) + J_n(x) + K_n(x).
\end{aligned}$$

We consider each of these terms in turn. By Hölder's inequality and Hardy's inequality (1.5), as $n \to \infty$,

$$\begin{aligned}
|I_n(x)| &\leq \frac{1}{2\pi}\|v_n'\|_2 \left(\int_{x-\pi}^{x+\pi} \left(\frac{\int_y^x \{f'(v_n(t)) - f'(v(t))\}\, v_n'(t)\,\mathrm{d}t}{\tan\frac{x-y}{2}}\right)^2 \mathrm{d}y\right)^{\frac{1}{2}} \\
&\leq \frac{2}{\pi}\|v_n'\|_2\|\{f'(v_n) - f'(v)\}v_n'\|_2 \\
&\leq \frac{2}{\pi}\|v_n'\|_2^2\|f'(v_n) - f'(v)\|_\infty \to 0,
\end{aligned}$$

since f' is continuous and (v_n') is bounded in Z_0. Also $K_n(x) \to 0$ as $n \to \infty$, since by (B.2-B.4),

$$y \mapsto \frac{f(v(x)) - f(v(y))}{\tan\frac{x-y}{2}}$$

is in $L^2_{2\pi}$ and $(v_n') \rightharpoonup v'$ in $L^2_{2\pi}$. Now

$$\begin{aligned}
J_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v_n'(y)\int_y^x\{f'(v(t)) - f'(v(x))\}\{v_n'(t) - v'(t)\}\,\mathrm{d}t}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&\quad + \frac{f'(v(x))}{2\pi}\int_{-\pi}^{\pi} \frac{v_n'(y)[\{v_n(x) - v(x)\} - \{v_n(y) - v(y)\}]}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{v_n'(y)\int_y^x\{f'(v(t)) - f'(v(x))\}\{v_n'(t) - v'(t)\}\,\mathrm{d}t}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&\quad + f'(v(x))(\mathcal{Q}(v_n)(x) - \mathcal{Q}(v)(x)) \\
&\quad + \frac{f'(v(x))}{2\pi}\int_{-\pi}^{\pi} \frac{\{v(x) - v(y)\}\{v_n'(y) - v'(y)\}}{\tan\frac{x-y}{2}}\,\mathrm{d}y \\
&= J_n^1(x) + J_n^2(x) + J_n^3(x).
\end{aligned} \tag{4.1}$$

We consider each of these terms in turn. By assumption, $(\mathcal{Q}(v_n)(x)) \to \mathcal{Q}(v)(x)$, so $(J_n^2(x)) \to 0$. Let, for $n \in \mathbb{N}$ and fixed x, $g_n : S^1 \to \mathbb{R}$ be given as follows: $g_n(x) = 0$ and for $y \neq x$

$$g_n(y) = \frac{\int_y^x (f'(v(t)) - f'(v(x)))(v_n'(t) - v'(t))\,\mathrm{d}t}{\tan\frac{x-y}{2}}.$$

Then if $y \neq x$,

$$\begin{aligned} |g_n(y)| &\leq \|f'\|_{C^{0,1}(\hat{J})} \frac{\int_y^x |v(t) - v(x)||v_n'(t) - v'(t)| \,\mathrm{d}t}{\tan \frac{x-y}{2}} \\ &\leq C \frac{\int_y^x |t - x|^{\frac{1}{2}} |v_n'(t) - v'(t)| \,\mathrm{d}t}{|x - y|}, \end{aligned}$$

for some constant C independent of y, since $f \in C^{1,1}_{\mathrm{loc}}(J)$ and $v \in W^{1,2}_{2\pi} \subset C^{0,1/2}_{2\pi}$ with $\mathcal{R}(v) \subset \hat{J}$. Hence by Hölder's inequality,

$$|g_n(y)| \leq C \frac{|x - y| \|v_n' - v'\|_2}{|x - y|} = C \|v_n' - v'\|_2,$$

so (g_n) is uniformly bounded as a function of y. Also, $(g_n(y)) \to 0$ since $(v_n') \rightharpoonup v'$ in $L^2_{2\pi}$; so $(g_n) \to 0$ in $L^2_{2\pi}$. An application of Hölder's inequality to (4.1) yields that $(J_n^1(x)) \to 0$.

By (B.2-B.4), the map

$$y \mapsto \frac{v(x) - v(y)}{\tan \frac{x-y}{2}}$$

is in $L^2_{2\pi}$, so using Hölder's inequality, and the fact that $(v_n') \rightharpoonup v'$ in $L^2_{2\pi}$, we have that $(J_n^3(x)) \to 0$, as required.

We have shown that, for a subsequence, $(\mathcal{F}(v_n)(x)) \to \mathcal{F}(v)(x)$ pointwise almost everywhere. Now by (B.2-B.5), $\|\mathcal{F}(v_n)\|_\infty \leq C \|v_n'\|_2^2$; as (v_n') is weakly convergent in $L^2_{2\pi}$, $\mathcal{F}(v_n)$ is uniformly bounded in $L^\infty_{2\pi}$. The result follows by dominated convergence. □

Corollary 4.3. *Let J be an open interval in $\mathbb{R}$ and $1 \leq p < \infty$. If $f \in C^{1,1}_{\mathrm{loc}}(J)$ then $\mathcal{F}$ is compact from $Z_1 \cap \Upsilon$ to the space of even $L^p_{2\pi}$-functions.*

Proof. Since $W^{1,2}_{2\pi}$ is a Hilbert space, each bounded sequence in $W^{1,2}_{2\pi}$ has a weakly convergent subsequence. Now apply Lemma 4.2. □

Theorem 4.4. *Let J be an open interval in $\mathbb{R}$ and suppose $f \in C^{1,1}_{\mathrm{loc}}(J)$ is bounded* *away from* 0. *If* $F : \Upsilon \cap Z_1 \to Z_0$ *is given by (3.3) then for all* $w \in \Upsilon \cap Z_1$, $\mathrm{d}F(w) : Z_1 \to Z_0$ *is Fredholm with index* 0.

Proof. Note first that $\Upsilon \cap Z_1$ is open in Z_1 since given $u \in \Upsilon \cap Z_1$, there exists $\varepsilon > 0$ such that $\operatorname{dist}(\mathcal{R}(u), \partial J) \geq \varepsilon$. Hence if $\tilde{u} \in Z_1$ with $\|\tilde{u} - u\|_{W^{1,2}_{2\pi}} < \varepsilon/\sqrt{2\pi}$ then

$$\|\tilde{u} - u\|_\infty \leq \sqrt{2\pi}\|\tilde{u} - u\|_{W^{1,2}_{2\pi}} \leq \varepsilon;$$

so $\mathcal{R}(\tilde{u}) \subset J$: i.e. $\tilde{u} \in \Upsilon \cap Z_1$.

It follows that we may differentiate F on $\Upsilon \cap Z_1$. Since for $w \in \Upsilon \cap Z_1$,

$$F(w) = 2f(w)(\mathcal{C}w' + w) - \mathcal{F}(w) + f(w)(1 - 2w) - f(0),$$

Note that since F and the map $w \mapsto f(w)(2\mathcal{C}w' + 1) - f(0)$ are Fréchet differentiable on $\Upsilon \cap Z_1$, so too is $\mathcal{F}$. It follows (by a standard calculation similar to those in Appendix C) that for all $h \in Z_1$,

$$\begin{aligned} \mathrm{d}F(w)h &= 2f(w)(\mathcal{C}h'+h)+2hf'(w)(\mathcal{C}w'+w)-\mathrm{d}\mathcal{F}(w)h-2hf(w)+hf'(w)(1-2w) \\ &= 2f(w)(\mathcal{C}h' + h) - \mathrm{d}\mathcal{F}(w)h + hf'(w)(2\mathcal{C}w' + 1) - 2hf(w). \qquad (4.2) \end{aligned}$$

By Lemma 4.1, the map $\Psi : Z_1 \to Z_0$ given by $\Psi(h) = f(w)(\mathcal{C}h' + h)$ is a homeomorphism. By Corollary 4.3, $\mathcal{F}$ is compact from $Z_1 \cap \Upsilon$ to Z_0; so for $w \in Z_1 \cap \Upsilon$, $\mathrm{d}\mathcal{F}(w)$ is compact from Z_1 to Z_0 (see Section 1.5).

We show that the remaining terms on the right-hand side of (4.2) are compact from Z_1 to Z_0. Let (h_n) be a bounded sequence in Z_1. Since Z_1 is compactly embedded in $L^\infty_{2\pi}$, $(h_n) \to h$ in $L^\infty_{2\pi}$ for a subsequence and some $h \in Z_1$, and

$$\|(h_n - h)(f'(w)(2\mathcal{C}w'+1) - 2f(w))\|_2 \leq \|h_n - h\|_\infty \|f'(w)(2\mathcal{C}w'+1) - 2f(w)\|_2 \to 0;$$

so $h \mapsto hf'(w)(2\mathcal{C}w' + 1) - 2hf(w)$ is compact from Z_1 into Z_0.

It follows that $\mathrm{d}F(w)$ is the sum of a compact operator and a homeomorphism from Z_1 into Z_0; so is Fredholm with index 0 (see Section 1.5). $\square$

For $S \subset \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, let

$$S_\lambda = \{y \in \mathbb{R} | (\lambda, y) \in S\}.$$

We now give a condition on f which ensures that hypothesis *III* in Theorem 1.25 holds.

Theorem 4.5. *Let $P \subset \mathbb{R}^2$ be open and let $f \in C^{1,1}_{\mathrm{loc}}(\overline{P})$ be such that $f \neq 0$ on $\overline{P}$. If U is a closed bounded set of solutions of (1.4) contained in $\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ then U is compact in $\mathbb{R} \times Z_1$.*

Proof. Let (λ_n, w_n) be a sequence in U. Then there exist $\lambda \in \mathbb{R}$ and $w \in Z_1$ such that for a subsequence $(\lambda_n) \to \lambda$ and $(w_n) \rightharpoonup w$ in $W^{1,2}_{2\pi}$; and for a further subsequence $(w_n) \to w$ in $L^\infty_{2\pi}$.

Now $\mathcal{R}(w) \subset \overline{P}_\lambda$ (otherwise there exists $\varepsilon > 0$ and $x \in S^1$ such that for all n, $|w(x) - w_n(x)| > \varepsilon$, which contradicts the fact that $(w_n) \to w$ in $L^\infty_{2\pi}$). We show that for a further subsequence, $(\mathcal{F}(\lambda_n, w_n)) \to \mathcal{F}(\lambda, w)$ in Z_0, where $\mathcal{F} : \{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\} \to Z_0$ is given by $\mathcal{F}(\lambda, w)(x) = f(\lambda, w(x))\mathcal{C}w'(x) - \mathcal{C}(f(\lambda, w)w')(x)$. We have that for $n \in \mathbb{N}$,

$$\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda, w) = \{\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda, w_n)\} + \{\mathcal{F}(\lambda, w_n) - \mathcal{F}(\lambda, w)\}.$$

By Lemma 4.2 we have that $\|\mathcal{F}(\lambda, w_n) - \mathcal{F}(\lambda, w)\|_2 \to 0$. If $g : \{(\lambda, \tilde{\lambda}, y) \in \mathbb{R}^3 | (\lambda, y), (\tilde{\lambda}, y) \in \overline{P}\} \to \mathbb{R}$ is given by $g(\lambda, \tilde{\lambda}, y) = f(\lambda, y) - f(\tilde{\lambda}, y)$ then by (2.6) and an argument similar to (B.2-B.5) we have that for all x,

$$\begin{aligned} &|\mathcal{F}(\lambda_n, w_n)(x) - \mathcal{F}(\lambda, w_n)(x)| \\ &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} \frac{g(\lambda_n, \lambda, w_n(x)) - g(\lambda_n, \lambda, w_n(y))}{\tan \frac{x-y}{2}} w_n'(y)\, \mathrm{d}y \right| \\ &\leq C \|g^{(0,0,1)}(\lambda_n, \lambda, \cdot)\|_\infty \|w_n'\|_2^2; \end{aligned}$$

but as $n \to \infty$,

$$\|g^{(0,0,1)}(\lambda_n, \lambda, w_n)\|_\infty = \|f^{(0,1)}(\lambda_n, w_n) - f^{(0,1)}(\lambda, w_n)\|_\infty \to 0.$$

Since $(w'_n) \rightharpoonup w'$ in $L^2_{2\pi}$, (w'_n) is bounded in $L^2_{2\pi}$; so $\|\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda, w_n)\|_\infty \to 0$. It follows that $\|\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda, w)\|_2 \to 0$.

Now by (2.19), for $n \in \mathbb{N}$,

$$\mathcal{C}w'_n = \frac{f(\lambda_n, 0) + \mathcal{F}(\lambda_n, w_n)}{2f(\lambda_n, w_n)} - \frac{1}{2}$$

(note that since $(\lambda_n, w_n) \in U$, $(\lambda_n, 0) \in P$); so for $m, n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{C}w'_n - \mathcal{C}w'_m &= \frac{f(\lambda_n, 0) + \mathcal{F}(\lambda_n, w_n)}{2f(\lambda_n, w_n)} - \frac{f(\lambda_m, 0) + \mathcal{F}(\lambda_m, w_m)}{2f(\lambda_m, w_m)} \\ &= \frac{f(\lambda_m, w_m)\{[f(\lambda_n, 0) - f(\lambda_m, 0)] + [\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda_m, w_m)]\}}{2f(\lambda_n, w_n)f(\lambda_m, w_m)} \\ &\quad + \frac{[f(\lambda_m, w_m) - f(\lambda_n, w_n)][f(\lambda_m, 0) + \mathcal{F}(\lambda_m, w_m)]}{2f(\lambda_n, w_n)f(\lambda_m, w_m)}. \end{aligned}$$

Now U is bounded in $\mathbb{R} \times W^{1,2}_{2\pi}$, so (λ_n) and $(\|w_n\|_\infty)$ are bounded; and $f(\lambda_n, w_n)$ is uniformly bounded away from 0. It follows that for some constants C_1, C_2 and C_3, independent of n and m,

$$\begin{aligned} \|\mathcal{C}w'_n - \mathcal{C}w'_m\|_2 &\le C_1|\lambda_n - \lambda_m| + C_2\|\mathcal{F}(\lambda_n, w_n) - \mathcal{F}(\lambda_m, w_m)\|_2 \\ &\quad + C_3\left(|\lambda_n - \lambda_m| + \|w_m - w_n\|_\infty\right) \to 0, \end{aligned}$$

as $n, m \to \infty$ since $f \in C^{0,1}_{\mathrm{loc}}(\overline{P})$; so $(\mathcal{C}w'_n)$ is Cauchy in $L^2_{2\pi}$, hence convergent. Hence as $(w'_n) \rightharpoonup w'$ in $L^2_{2\pi}$, we have by the theorem of M.Riesz (Theorem 1.7 (A)) that $w'_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} w'_n(t)\ dt - \mathcal{C}^2 w'_n \to w'$ in $L^2_{2\pi}$ [note that $\int_{-\pi}^{\pi} w'_n(t)\ dt = 0$ for all n]; and so $(w_n) \to w$ in Z_1.

Since U is closed, it now follows that $(\lambda, w) \in U$. The compactness of U

follows. $\square$

We are now in a position to apply the real-analytic bifurcation theorem (Theorem 1.25) to (1.4)

Theorem 4.6. *Let P be an open neighbourhood of $(\lambda^*, 0)$ and let $f \in C^{1,1}_{\mathrm{loc}}(\overline{P})$ be such that f is real-analytic and bounded away from 0 on P.*

Suppose $(\lambda^*, 0)$ is either a Crandall-Rabinowitz type bifurcation point or a double bifurcation point of (1.4) (satisfying Theorem 3.13), and let $\mathfrak{B}$ be a curve of nontrivial solutions of (1.4) which bifurcates from $(\lambda^*, 0)$. Let $\mathfrak{B}^+ = \{\mathcal{B}(t) | t \in [0, \varepsilon)\}$ (see Theorem 3.7 for notation).

If $(\lambda^*, 0)$ is a bifurcation point of Crandall-Rabinowitz type, suppose also that there exists a neighbourhood of 0 on which $\mu' \not\equiv 0$.

Then there exists a continuous locally injective extension, $\tilde{\mathfrak{B}}$ of $\mathfrak{B}^+$, as follows:

(i) $\tilde{\mathfrak{B}} = \{\mathcal{B}(t) | t \in [0, \infty)\}$, *where* $\mathcal{B} : [0, \infty) \to \{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$.

(ii) $F(\mathcal{B}(t)) = 0$ *for all* $t \geq 0$.

(iii) The set $\{t \geq 0 | \mathcal{N}(\partial^{0,1} F(\mathcal{B}(t))) \neq \{0\}\}$ *has no accumulation points.*

(iv) For each $t^* > 0$ there exists $\delta^* \in (0, t^*)$ and a real-analytic map $\sigma^* : (-\delta^*, \delta^*) \to \{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ such that

$$\{\mathcal{B}(t) || t - t^*| < \delta^*\} = \{\sigma^*(t) || t| < \delta^*\}.$$

(v) One of the following occurs:

a. $\|\mathcal{B}(t)\|_{\mathbb{R} \times W^{1,2}_{2\pi}} \to \infty$ *as* $t \to \infty$.

b. $\mathcal{B}(t)$ *approaches* $\partial\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ *as* $t \to \infty$.

c. $\tilde{\mathfrak{B}}$ *is a closed loop: i.e. for some* $T > 0$, $\tilde{\mathfrak{B}} = \{\mathcal{B}(t) | t \in [0, T]\}$, *where* $\mathcal{B}(T) = \mathcal{B}(0) = (\lambda^*, 0)$.

If instead of requiring that f is bounded away from 0 on P we require only that $f \neq 0$ on $\overline{P}$ then we have that there exists a continuous locally injective extension, $\tilde{\mathfrak{B}}$ of $\mathfrak{B}^+$ such that (i) and (ii) hold, and (v) holds, with a. replaced by

a'. $\mathfrak{B}$ contains a sequence (λ_n, w_n) such that $\|(\lambda_n, w_n)\|_{\mathbb{R}\times W^{1,2}_{2\pi}} \to \infty$.

Proof. Suppose first that f is bounded away from 0 on P. Since f is real-analytic, so too is F, by Lemma 3.1. We check that conditions *I-III* in the real-analytic bifurcation theorem (Theorem 1.25) are satisfied on P.

By Theorem 4.4, $\partial^{0,1}F(\lambda, w)$ is Fredholm index 0 for all $(\lambda, w) \in \mathbb{R} \times Z_1$ with $\mathcal{R}(w) \subset P_\lambda$, so condition *I* holds.

ε may be chosen such that *II (a)* holds, since

$$\mathrm{d}W(t)1 = v^* + \partial^{1,0}\gamma(\lambda^* + \mu(t), tv^*)\mu'(t) + \partial^{0,1}\gamma(\lambda^* + \mu(t), tv^*)v^*,$$

$v^* \neq 0$, $\partial^{1,0}\gamma(\lambda^*, 0) = 0$, $\partial^{0,1}\gamma(\lambda^*, 0) = 0$, and γ and μ are C^1.

If $(\lambda^*, 0)$ is a bifurcation point of Crandall-Rabinowitz type then it is a hypothesis that there exists a neighbourhood of 0 on which $\mu' \equiv \Lambda' \not\equiv 0$. If $(\lambda^*, 0)$ satisfies Theorem 3.13 then by (3.17) $\mu'(0) \neq 0$. Hence *II (b)* holds.

By Theorem 4.5 if U is a closed bounded subset of the set of solutions of (1.4) in $\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ then U is compact in $\mathbb{R} \times Z_1$; so condition *III* holds.

In the case where only $f \neq 0$ on $\overline{P}$ let, for $n \in \mathbb{N}$ with $n > \lambda^*$, $P^n = P \cap \{z \in \mathbb{R}^2 | |z| < n\}$. Then f is bounded away from 0 on each P^n and $(\lambda^*, 0) \in P^n$. It follows that for each n there exists a continuous locally injective extension $\mathfrak{B}_n = \{\mathcal{B}_n(t) | t \in [0, \infty)\} \subset \{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P^n_\lambda\}$ of $\mathfrak{B}^+$ such that *(i)-(v)* hold with P^n for P, $\mathfrak{B}_n$ for $\tilde{\mathfrak{B}}$ and $\mathcal{B}_n$ for $\mathcal{B}$.

Suppose for all n that *(v) c* does not occur, and $\mathcal{B}_n(t)$ does not approach $\partial\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ as $t \to \infty$.

By *(iv)*, if $m \geq n > \lambda^*$ then $\mathfrak{B}_n \subset \mathfrak{B}_m$. It follows that there exists a common continuous reparametrization $\mathcal{B} : [0,\infty) \to \mathbb{R} \times Z_1$ of each $\mathfrak{B}_n$ such that $\mathcal{B}_n([0,\infty)) = \mathcal{B}([0,n))$. We consider the points $\mathcal{B}(n) = (\lambda_n, w_n)$. Since for all n, $\mathcal{B}_n(t)$ does not approach $\partial\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ as $t \to \infty$, we have that $\mathcal{R}(w_n) \cap \partial\{z \in \mathbb{R}^2 || z| < n\}_{\lambda_n} \neq \emptyset$. Suppose $|\lambda_n|$ is bounded (otherwise $(\|(\lambda_n, w_n)\|_{\mathbb{R}\times W^{1,2}_{2\pi}}) \to \infty$ for a subseqence). Then for some $\lambda_0 \geq 0$, $\|w_n\|_{W^{1,2}_{2\pi}} \geq \frac{1}{\sqrt{2\pi}}\|w_n\|_\infty \geq \sqrt{\frac{n^2-\lambda_0^2}{2\pi}} \to \infty$. □

Remark 4.7. If $(\lambda^*, 0)$ is a bifurcation point of Crandall-Rabinowitz type then sufficient conditions for the condition $\mu' \not\equiv 0$ are given in Theorems 3.9 (if $q^* \neq 0$) and 3.11 (if $q^* = 0$). ■

4.2 Entrapment of curves of solutions

In this section we find conditions on f and P which prevent $\mathcal{B}(t)$ approaching $\partial\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P_\lambda\}$ as $t \to \infty$, and prevent $\lambda \to \infty$ and $\|w\|_{W^{1,2}_{2\pi}} \to \infty$ in *(v)a*. We give two sets of conditions which achieve such "entrapment". The conditions in the following theorem do not achieve entrapment - they leave open the possibility that $\tilde{\mathfrak{B}}$ "escapes" through one of two points. The theorem illustrates some of the difficulties which arise when f vanishes.

Theorem 4.8. *Let $\lambda_1, \dots, \lambda_4 \in \mathbb{R}$ with $\lambda_1 < \cdots < \lambda_4$, and suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is real-analytic on $(\lambda_1, \lambda_4) \times \mathbb{R}$. Let $O = \{(\lambda, y) \in (\lambda_1, \lambda_4) \times \mathbb{R} | f(\lambda, y) > 0\}$, and suppose that $O \cap ([\lambda_2, \lambda_3] \times \mathbb{R}) = ([\lambda_2, \lambda_3] \times \mathbb{R}) \setminus \{(\lambda_2, 0), (\lambda_3, 0)\}$.*

If $\lambda^* \in (\lambda_2, \lambda_3)$ is such that $(\lambda^*, 0)$ is a bifurcation point of (1.4) of Crandall-Rabinowitz type, and the function $\mu' \not\equiv 0$ in a neighbourhood of 0, then there exists a continuous locally injective extension, $\tilde{\mathfrak{B}}$ of $\mathfrak{B}^+$, such that $f(\Lambda(t), W(t)(s)) > 0$ and $F(\mathcal{B}(t)) = 0$ for all $t \geq 0$ and $s \in S^1$, and one of the following holds:

A. $\tilde{\mathfrak{B}}$ forms a closed loop.

B. $\tilde{\mathfrak{B}}$ contains a sequence (λ_n, w_n) such that $\|w_n\|_{W^{1,2}_{2\pi}} \to \infty$.

C. $\tilde{\mathfrak{B}}$ *contains a sequence* (λ_n, w_n) *such that either* $(\lambda_n, w_n) \to (\lambda_2, 0)$ *or* $(\lambda_n, w_n) \to (\lambda_3, 0)$ *in* $\mathbb{R} \times L^{\infty}_{2\pi}$.

If f is bounded away from 0 on $[\lambda_2, \lambda_3] \times \mathbb{R}$ away from $\{(\lambda_2, 0), (\lambda_3, 0)\}$ then B. may be replaced with

B′. $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$

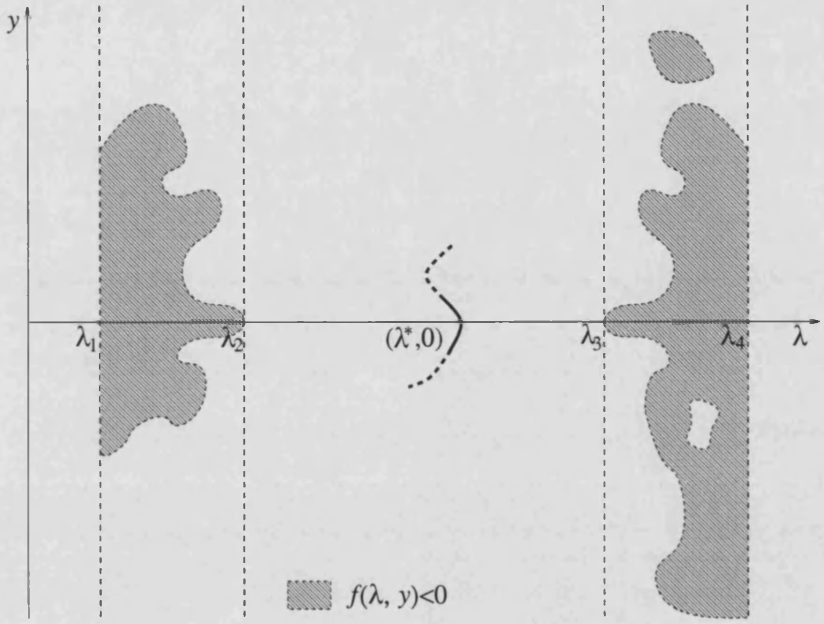

Figure 4-1: Diagram for Theorem 4.8. Note that the curve bifurcating from $(\lambda^*, 0)$ is for illustration only

Proof. Let, for $n \in \mathbb{N}$, $P^n = \{(\lambda, y) \in (\lambda_1, \lambda_4) \times \mathbb{R} | f(\lambda, y) > 1/n\}$. Then, O and each P^n is open, $f \neq 0$ on each $\overline{P^n}$ and for sufficiently large n ($n > n_0$, say), $(\lambda^*, 0) \in P^n$; so Theorem 4.6 applies.

Let $n > n_0$. Then there exists a continuous extension $\mathfrak{B}_n = \{\mathcal{B}_n(t) | t \in [0, \infty)\} \subset \{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P^n_\lambda\}$ of $\mathfrak{B}^+$ such that either $\mathfrak{B}_n$ forms a closed loop or contains a sequence (λ_n, w_n) such that $\|(\lambda_n, w_n)\|_{\mathbb{R} \times W^{1,2}_{2\pi}} \to \infty$ [or $\|\mathcal{B}(t)\|_{\mathbb{R} \times W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$ if f satisfies the stronger set of hypotheses in the theorem]; or as $t \to \infty$, $\mathcal{B}_n(t)$ approaches $\partial\{(\lambda, w) \in \mathbb{R} \times Z_1 | \mathcal{R}(w) \subset P^n_\lambda\}$.

We show that $\mathfrak{B}_n \subset \{(\lambda, w) \in (\lambda_2, \lambda_3) \times Z_1 | \mathcal{R}(w) \subset P_\lambda^n\}$. Suppose otherwise. Since $\mathcal{B}_n$ is continuous, we have that (without loss of generality) there exists $w \in Z_1$ with $\mathcal{R}(w) \subset P_{\lambda_2}^n$ such that $(\lambda_2, w) \in \mathfrak{B}_n$. Now $f(\lambda_2, w) \geq 0$ almost everywhere, so by Theorem 2.19 w satisfies the Bernoulli condition (Theorem 2.19 (β)); and $f(\lambda_2, w(t)) = 0$ almost everywhere. Hence, as $f(\lambda_2, y) = 0$ if and only if $y = 0$, we have that $w \equiv 0$; but $0 \notin P_{\lambda_2}^n$, a contradiction.

Suppose that neither A nor B occurs (so neither A nor B occurs for $\mathfrak{B}_n$ $(n \geq n_0)$). Then each $\mathfrak{B}_n$ satisfies Theorem 4.6 (iv). Hence if $m \geq n > n_0$ then $\mathfrak{B}_n \subset \mathfrak{B}_m$. It follows that there exists a common continuous reparametrization $\mathcal{B} : [0, \infty) \to [\lambda_2, \lambda_3] \times Z_1$ of each $\mathfrak{B}_n$ $(n > n_0)$ such that $\mathcal{B}_n([0, \infty)) = \mathcal{B}([0, n))$. We show that there is a subsequence (λ_n, w_n) of $(\mathcal{B}(n))$ such that either $\|(\lambda_n - \lambda_2, w_n)\|_{\mathbb{R} \times L_{2\pi}^\infty} \to 0$ or $\|(\lambda_n - \lambda_3, w_n)\|_{\mathbb{R} \times L_{2\pi}^\infty} \to 0$.

Let $\mathcal{B}(n) = (\lambda_n, w_n)$. Then by Theorem 4.6 *(v) b*, $\operatorname{dist}(\lambda_n, \{\lambda_2, \lambda_3\}) \to 0$. Hence there exists a subsequence such that either $(\lambda_n) \to \lambda_2$ or $(\lambda_n) \to \lambda_3$. Suppose that $(\lambda_n) \to \lambda_2$. Also (w_n) is bounded in $W_{2\pi}^{1,2}$ (otherwise B occurs), so for a subsequence and some $w \in W_{2\pi}^{1,2}$, $(w_n) \rightharpoonup w$ in $W_{2\pi}^{1,2}$ and $(w_n) \to w$ in $L_{2\pi}^\infty$. Now

$$\begin{aligned}
&f(\lambda_n, w_n)(1 + \mathcal{C}w_n') + \mathcal{C}(f(\lambda_n, w_n)w_n')) \\
&\quad = [f(\lambda_n, w_n) - f(\lambda_2, w_n)](1 + \mathcal{C}w_n') + \mathcal{C}([f(\lambda_n, w_n) - f(\lambda_2, w_n)]w_n') \\
&\qquad + f(\lambda_2, w_n)(1 + \mathcal{C}w_n') + \mathcal{C}(f(\lambda_2, w_n)w_n').
\end{aligned} \tag{4.3}$$

It is easily seen, using the continuity of f, the convergence of (λ_n), the boundedness in $L_{2\pi}^2$ of (w_n') and the theorem of M. Riesz (Theorem 1.7 (A)) that the terms on the first line of the right-hand side of (4.3) converge strongly in $L_{2\pi}^2$ to 0.

Since $(w_n) \rightharpoonup w$ in $W_{2\pi}^{1,2}$, we have by the theorem of M.Riesz (Theorem 1.7 (A)) that $(\mathcal{C}w_n') \rightharpoonup \mathcal{C}w'$ in $L_{2\pi}^2$ [recall that the image of a weakly convergent sequence under a bounded linear operator converges weakly to the image of the weak limit of the sequence under the same operator]. Now

$$f(\lambda_2, w_n)(1+\mathcal{C}w_n') = [f(\lambda_2, w_n) - f(\lambda_2, w)](1+\mathcal{C}w_n') + f(\lambda_2, w)(1+\mathcal{C}w_n'). \quad (4.4)$$

Since $f(\lambda_2, w_n) \to f(\lambda_2, w)$ in $L^\infty_{2\pi}$, the first term on the right-hand side of (4.4) converges strongly to 0 in $L^2_{2\pi}$; the second converges weakly to $f(\lambda_2, w)(1+\mathcal{C}w')$ in $L^2_{2\pi}$, since $f(\lambda_2, w) \in L^2_{2\pi}$. Hence $f(\lambda_2, w_n)(1+\mathcal{C}w_n') \rightharpoonup f(\lambda_2, w)(1+\mathcal{C}w')$ in $L^2_{2\pi}$. Similarly, $\mathcal{C}(f(\lambda_2, w_n)w_n') \rightharpoonup \mathcal{C}(f(\lambda_2, w)w')$ in $L^2_{2\pi}$.

It follows that in $L^2_{2\pi}$,

$$f(\lambda_n, w_n)(1+\mathcal{C}w_n') + \mathcal{C}(f(\lambda_n, w_n)w_n')) \rightharpoonup f(\lambda_2, w)(1+\mathcal{C}w') + \mathcal{C}(f(\lambda_2, w)w'));$$

but $f(\lambda_n, w_n)(1+\mathcal{C}w_n') + \mathcal{C}(f(\lambda_n, w_n)w_n')) = f(\lambda_n, 0) \to f(\lambda_2, 0)$; so $f(\lambda_2, w)(1+\mathcal{C}w') + \mathcal{C}(f(\lambda_2, w)w')) = f(\lambda_2, 0) = 0$, and $w \equiv 0$ by the argument above.

Hence $\tilde{\mathfrak{B}} := \cup_{n>n_0}\mathfrak{B}_n$ contains a sequence (λ_n, w_n) such that $(\lambda_n, w_n) \to (\lambda_2, 0)$ in $\mathbb{R}\times L^\infty_{2\pi}$. The same argument covers the case $(\lambda_n) \to \lambda_3$, as required. □

Remark 4.9. At the points $(\lambda_2, 0)$ and $(\lambda_3, 0)$, we have $f = 0 = f^{(0,1)}$, so that $\partial^{0,1}F(\lambda_2, 0)$ and $\partial^{0,1}F(\lambda_3, 0)$ are the zero operator from Z_1 into Z_0. In particular, $\partial^{0,1}F(\lambda_2, 0)$ and $\partial^{0,1}F(\lambda_3, 0)$ are not Fredholm. ■

Example 4.10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = \left(1 + 4\left((\lambda+y)^2 - 1\right)^2\right)\left(1 + y^2 - \lambda^2\right).$$

Then f is real-analytic on $\mathbb{R}^2$ and $f(\lambda, y) > 0$ if and only if (λ, y) lies between the two branches of the hyperbola $\lambda^2 - y^2 = 1$, which cross the λ axis at $(-1, 0)$ and $(1, 0)$. We show that there is a bifurcation of nonconstant even solutions of (1.4) of Crandall-Rabinowitz type at some point $(\lambda^*, 0)$, where $\lambda^* \in (-1, 1)$.

By direct calculation it is found that

$$q(\lambda) = -\frac{8\lambda(\lambda^2-1)}{1+4(\lambda^2-1)^2}.$$

Since $1+4(\lambda^2-1)^2 \neq 0$ for all λ, q is continuous. Now $q(1/2) = 12/13 < 1$ and $q(3/5) = 1920/1649 > 1$; so by the intermediate value theorem there exists $\lambda^* \in (1/2, 3/5)$ such that $q^* = q(\lambda^*) = 1$. By inspection, for all $\lambda \in [1/2, 3/5]$, $q'(\lambda) \in [2.3, 2.5]$. It follows from Theorem 4.6 that a unique branch $\mathfrak{B}$ of nonconstant even solutions of (1.4) bifurcates from the branch of trivial solutions at $(\lambda^*, 0)$. Also, by inspection, for $\lambda \in (1/2, 3/5)$, $B(\lambda) \in [-6, -5]$; so in particular by Theorem 3.7, $\mu''(0) \neq 0$, so there exists a neighbourhood of 0 on which $\mu' \not\equiv 0$. It follows that f and λ^* satisfy the hypotheses of Theorem 4.8; so $\tilde{\mathfrak{B}}$ must either contain a sequence converging in $\mathbb{R} \times L^\infty_{2\pi}$ to $(-1, 0)$ or $(1, 0)$, form a closed loop or be such that $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$. ∎

The next set of conditions relies on the following result, due to Toland [21].

Theorem 4.11. *Let $f \in C^1(J)$ and $c \in \mathbb{R} \setminus \{0\}$. Suppose $w \in W^{1,1}_{2\pi}$ is a nonconstant solution of (1.1) which satisfies the Bernoulli condition (Theorem 2.19 (β)). Then f/c is strictly decreasing on an interval in $\mathcal{R}(w)$.*

Proof. Suppose f and c satisfy the hypotheses of the theorem, and let $w \in W^{1,1}_{2\pi}$ be a solution of (1.1) which satisfies the Bernoulli condition. Then $f(w(x))\{w'(x)^2 + (1+\mathcal{C}w'(x))^2\} = c$ for almost every x. For such x,

$$\begin{aligned} f(w(x))\{w'(x)^2 + (\mathcal{C}w'(x))^2\} &= c - f(w(x)) - 2f(w(x))\mathcal{C}w'(x) \\ &= \mathcal{C}(f(w)w')(x) - f(w(x))\mathcal{C}w'(x) = -\mathcal{F}(w)(x), \end{aligned} \tag{4.5}$$

since w satisfies (1.1). By the Bernoulli condition, $f(w(x))/c > 0$; so by (4.5), $\mathcal{F}(w)(x)/c \leq 0$.

Now by Lemma 2.16, if f/c is monotone increasing on $\mathcal{R}(w)$ then for almost all x, $\mathcal{F}(w)(x)/c \geq 0$. Hence $\mathcal{F}(w)(x) = 0$ almost everywhere. Equation (4.5), together with the fact that $f(w(x))/c > 0$ almost everywhere, now gives that $w'(x) = 0$ for almost every x: i.e. w is a constant function.

Hence if w is nonconstant then f/c is not monotone increasing on $\mathcal{R}(w)$: i.e. (since $f \in C^1(J)$) f/c is strictly decreasing on an interval in $\mathcal{R}(w)$. $\square$

Theorem 4.12. *Let $\lambda_1, \ldots, \lambda_4 \in \mathbb{R}$ with $\lambda_1 < \cdots < \lambda_4$ and suppose that $f : [\lambda_1, \lambda_4] \times \mathbb{R} \to \mathbb{R}$ is bounded away from 0, f is real-analytic and*

$$\left\{ (\lambda, y) \in [\lambda_1, \lambda_4] \times \mathbb{R} \middle| \frac{f^{(0,1)}(\lambda, y)}{f(\lambda, 0)} < 0 \right\} = \Omega \subset (\lambda_2, \lambda_3) \times \mathbb{R}$$

where $(\lambda^*, 0) \in \Omega$. If $(\lambda^*, 0)$ is a bifurcation point of (1.4) of Crandall-Rabinowitz type and the function $\mu' \not\equiv 0$, then either the curve $\tilde{\mathfrak{B}}$ found in Theorem 4.6 forms a closed loop or $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$.

Proof. Note first that f, $(\lambda_1, \lambda_4) \times \mathbb{R}$ and $(\lambda^*, 0)$ satisfy the hypotheses of Theorem 4.6, and that $q^* \neq 0$ since $f^{(0,1)}(\lambda^*, 0)/f(\lambda^*, 0) < 0$. Since f is bounded away from 0 on $(\lambda_1, \lambda_4) \times \mathbb{R}$, the curve $\tilde{\mathfrak{B}}$ satisfies *(i)-(v)* in Theorem 4.6. We show that *(v) b* cannot occur, and that if *(v) a* occurs then $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$.

There is no nonconstant solution (λ, w) of (1.4) with $\lambda \in [\lambda_1, \lambda_4] \setminus [\lambda_2, \lambda_3]$, since for any solution (λ, w) with $\lambda \in [\lambda_1, \lambda_4] \setminus [\lambda_2, \lambda_3]$, $f^{(0,1)}(\lambda, w(x))/f(\lambda, 0) \geq 0$ for all x, so by Theorem 4.11, w is constant.

We show that $\tilde{\mathfrak{B}} \subset [\lambda_2, \lambda_3] \times W^{1,2}_{2\pi}$. Suppose for a contradiction that for some $\hat{\lambda} \in [\lambda_1, \lambda_4] \setminus [\lambda_2, \lambda_3]$, there exists $w \in Z_1$ such that $(\hat{\lambda}, w) = \mathcal{B}(\hat{t}) \in \tilde{\mathfrak{B}}$. By Theorem 4.6 *iv*, for each $s \in (0, \hat{t}]$ there exists $\delta_s \in (0, s)$ and a real-analytic map $\sigma_s : (-\delta_s, \delta_s) \to I \times \{w \in Z_1 | \mathcal{R}(w) \subset J\}$ such that

$$\{\mathcal{B}(t) | |t - s| < \delta_s\} = \{\sigma_s(t) | |t| < \delta_s\}.$$

Let $t_0 > 0$ be close to 0, and let $\{t_0, \ldots, t_n\}$ $(t_n = \hat{t})$ be a partition of $[t_0, \hat{t}]$ such that $t_0 < \cdots < t_n$ and

$$\{\mathcal{B}(t) | t \in [t_0, \hat{t}]\} \subset \bigcup_{i=0}^{n} \{\sigma_{t_i}(t) | |t| < \delta_{t_i}\}$$

(this is possible since $[t_0, \hat{t}]$ is compact, so the cover $\bigcup_{s \in [t_0, \hat{t}]} \{\sigma_s(t) | |t| < \delta_s\}$ of $\{\mathcal{B}(t) | t \in [t_0, \hat{t}]\}$ has a finite subcover). Note that for $i \in \{1, \ldots, n\}$,

$$\{\sigma_{t_{i-1}}(t) | |t| < \delta_{t_{i-1}}\} \cap \{\sigma_{t_i}(t) | |t| < \delta_{t_i}\} \neq \emptyset.$$

Let, for $i \in \{0, \ldots, n\}$, $\sigma_{t_i}(t) = (\Lambda_i(t), W_i(t))$, and let $\psi_i : (-\delta_{t_i}, \delta_{t_i}) \to \mathbb{R}$ be given by $\psi_i(t) = \langle \Phi_{q^*}, W_i(t) \rangle_2$. Since each σ_{t_i} is real-analytic, so too is W_i, and hence ψ_i also.

The remarks in the second paragraph of this proof show that for t in an open interval, $W_n(t)$ is a constant function; and so $\psi_n(t) = 0$ on an open interval, since $q^* \neq 0$. Since ψ_n is real-analytic, it follows that $\psi_n \equiv 0$. Now for each t in an open interval, there exists $t' \in (-\delta_{t_n}, \delta_{t_n})$ such that $\sigma_{t_{n-1}}(t) = \sigma_{t_n}(t')$, so $\psi_{n-1}(t) = \psi_n(t') = 0$. It follows that ψ_{n-1} is zero on an open interval, so is everywhere zero, since it is real-analytic. By an inductive argument it follows that each of $\psi_0, \ldots, \psi_n$ are everywhere zero. However, by (3.11) $\langle \Phi_{q^*}, W_0(0) \rangle_2 = \langle \Phi_{q^*}, W(t_0) \rangle_2 \neq 0$ if t_0 is sufficiently small. This is a contradiction.

It follows from Theorem 4.6 v that either $\tilde{\mathfrak{B}}$ forms a closed loop or $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$. □

Example 4.13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = 1 + 4\mathrm{e}^{3y(\lambda^2 - 1)}$$

Then f is real-analytic and bounded away from 0. Also, $f^{(0,1)}(\lambda, y) = 12(\lambda^2 - 1)\mathrm{e}^{3y(\lambda^2-1)}$, so $f^{(0,1)}(\lambda, y) < 0$ if and only if $\lambda \in (-1, 1)$. For all λ we have $q(\lambda) = 6(1 - \lambda^2)/5$ so $q(\lambda) = 1$ if and only if $\lambda = \pm 1/\sqrt{6}$. Now $q'(\lambda) = -12\lambda/5$, so

$q'(\pm 1/\sqrt{6}) = \mp 2\sqrt{6/5} \neq 0$. Hence by Theorem 3.7, $(\pm\sqrt{1/6}, 0)$ are bifurcation points of (1.4) of Crandall-Rabinowitz type; and a unique real-analytic curve $\mathfrak{B}$ of nonconstant solutions of (1.4) bifurcates from the line of trivial solutions at $(\sqrt{1/6}, 0)$. Also, $B(1/\sqrt{6}) = -105/4 \neq 0$, so by Theorem 3.9, $\mu''(0) \neq 0$; so $\mu' \not\equiv 0$ in a neighbourhood of 0. It follows that f and $1/\sqrt{6}$ satisfy the hypotheses for Theorem 4.12 to hold; so either $\tilde{\mathfrak{B}}$ must form a closed loop or $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$. Similarly for the curve bifurcating from $(-1/\sqrt{6}, 0)$. ∎

A refinement of Theorem 4.12 is the following:

Theorem 4.14. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be real-analytic with $|f| \geq k > 0$ everywhere, and suppose that*

$$\frac{f^{(0,1)}(\lambda, y)}{f(\lambda, 0)} < 0$$

on a set $\Omega \subset \mathbb{R}^2$ containing $(\lambda^*, 0)$. Suppose that for all λ for which $\Omega_\lambda \neq \emptyset$, $y_\lambda := \sup\{|y| \,|\, y \in \Omega_\lambda\} < \infty$.

Let

$$\Omega' = \left\{ (\lambda, y) \in \mathbb{R}^2 \,\middle|\, \frac{f^{(0,1)}(\lambda, y)}{f(\lambda, 0)} < 0 \right\} \setminus \Omega$$

and suppose that either $\Omega' = \emptyset$ or for each λ for which $\Omega_\lambda \neq \emptyset$, $h_\lambda := \mathrm{dist}(\{\lambda\} \times \overline{\Omega}_\lambda, \Omega') > \pi\sqrt{\frac{|f(\lambda,0)| - k}{2k}}$.

1. *If $(\lambda^*, 0)$ is a bifurcation point of (1.4) of Crandall-Rabinowitz type and $\mu' \not\equiv 0$ then either $\tilde{\mathfrak{B}}$ forms a closed loop or $\Lambda(t) \to \infty$ as $t \to \infty$.*

   If Ω is bounded then only the first of these possibilities is allowed.

2. *If $(\lambda^*, 0)$ is a double bifurcation point of (1.4) (satisfying Theorem 3.13) and Ω is bounded, then the set of nontrivial solutions of (1.4) is at least triply connected.*

Proof. Suppose first that $f \geq k$ everywhere, and let $(\lambda, w) \in \mathbb{R} \times W^{1,2}_{2\pi}$ be a solution of (1.4). Then by Theorem 2.21 *(a)* and Theorem 2.19 (β),

$$f(\lambda, 0) \geq k\{w'(t)^2 + (1 + \mathcal{C}w'(t))^2\} = k\{w'(t)^2 + 1 + 2\mathcal{C}w'(t) + (\mathcal{C}w'(t))^2\} \quad (4.6)$$

for almost every t. Integrating we obtain that

$$2\pi(f(\lambda, 0) - k) - 2\int_{-\pi}^{\pi} \mathcal{C}w'(t)\ \mathrm{d}t \geq k\{\|w'\|_2^2 + \|\mathcal{C}w'\|_2^2\}, \quad (4.7)$$

so that by the theorem of M. Riesz (Theorem 1.7 (A)),

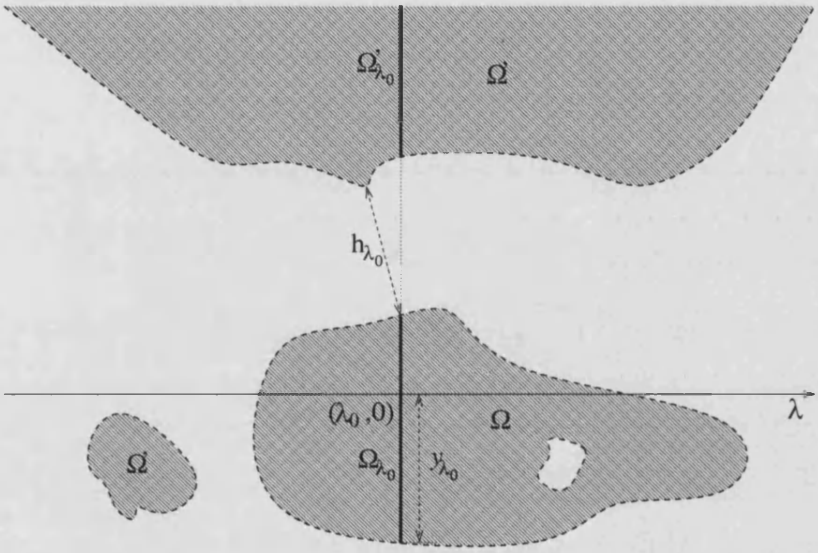

Figure 4-2: Diagram for Theorem 4.14

$$\|w'\|_2^2 \leq \pi \frac{|f(\lambda, 0)| - k}{k} \quad (4.8)$$

(note that $\mathcal{C}w'$ has zero mean, and recall that $\mathcal{C}^2 w' = -w'$). A similar calculation shows that (4.8) holds if $f \leq -k$ also.

By (1.11) and (4.8),

$$\mathrm{Amp}(w) \leq \pi\sqrt{\frac{|f(\lambda, 0)| - k}{2k}}. \quad (4.9)$$

Suppose now $\Omega' \neq \emptyset$ and let (λ, w) be a solution with $\mathcal{R}(w) \cap \overline{\Omega}_\lambda \neq \emptyset$. We show that $\mathrm{dist}(\{\lambda\} \times \mathcal{R}(w), \Omega') > 0$. Suppose otherwise. Then for each $\varepsilon > 0$ there exists $y \in \mathcal{R}(w)$ such that $\mathrm{dist}((\lambda, y), \Omega') < \varepsilon$. Hence for all $\varepsilon > 0$,

$$h_\lambda = \mathrm{dist}(\{\lambda\} \times \overline{\Omega}_\lambda, \Omega') < \mathrm{Amp}(w) + \varepsilon :$$

i.e. $h_\lambda \leq \mathrm{Amp}(w)$. However it follows from the hypotheses of the theorem and from (4.9) that $h_\lambda > \mathrm{Amp}(w)$: a contradiction. Hence $\mathrm{dist}(\{\lambda\} \times \mathcal{R}(w), \Omega') > 0$, as required.

We now show that for each $(\lambda, w) \in \tilde{\mathfrak{B}}$, $\mathcal{R}(w) \cap \overline{\Omega}_\lambda \neq \emptyset$. Suppose by way of contradiction that for some positive t and ε, $\mathcal{R}(W(t)) \cap \overline{\Omega}_{\Lambda(t)} \neq \emptyset$, and that if $t < s < t + \varepsilon$ then $\mathcal{R}(W(s)) \cap \overline{\Omega}_{\Lambda(s)} = \emptyset$. Let

$$d = \mathrm{dist}(\{\Lambda(t)\} \times \mathcal{R}(W(t)), \Omega') > 0.$$

The map $s \mapsto \mathcal{B}(s)$ has, at each point s, a local real-analytic reparametrization, so is continuous. Hence for some $\varepsilon' > 0$, if $|s - t| < \varepsilon'$ then $|\Lambda(t) - \Lambda(s)| < d/2$ and

$$\|W(t) - W(s)\|_\infty \leq \sqrt{2\pi}\|W(t) - W(s)\|_{W^{1,2}_{2\pi}} < \frac{d}{2}.$$

It follows that if $|s - t| < \varepsilon'$ then $\mathcal{R}(W(s)) \cap \overline{\Omega}'_{\Lambda(s)} = \emptyset$; and hence if $t < s < t + \min\{\varepsilon, \varepsilon'\}$ then $\mathcal{R}(W(s)) \cap (\Omega_{\Lambda(s)} \cup \Omega'_{\Lambda(s)}) = \emptyset$, so by Theorem 4.11 $W(s)$ is a constant function. The required contradiction follows from an argument similar to the one in the proof of Theorem 4.12 (note that, as in Theorem 4.12, $q^* \neq 0$).

It now follows from (4.8-4.9) that (whether or not $\Omega' = \emptyset$) if $(\lambda, w) \in \tilde{\mathfrak{B}}$, then for almost every x, $|w(x)| \leq \pi\sqrt{\frac{|f(\lambda,0)|-k}{2k}} + y_\lambda$ and

$$\|w\|^2_{W^{1,2}_{2\pi}} \leq 2\pi \left(\pi\sqrt{\frac{|f(\lambda, 0)| - k}{2k}} + y_\lambda\right)^2 + \pi\frac{|f(\lambda, 0)| - k}{k}. \tag{4.10}$$

Since f is real-analytic, $|f(\lambda, 0)| < \infty$ for all λ; also $y_\lambda < \infty$ for all λ. Hence in particular $f(\lambda, 0)$ and y_λ are bounded on bounded intervals, so by (4.10) we have that if $\|W(t)\|_{W^{1,2}_{2\pi}} \to \infty$ as $t \to \infty$ then $|\Lambda(t)| \to \infty$ as $t \to \infty$ also.

Hence by Theorem 4.6, $\tilde{\mathfrak{B}}$ either forms a closed loop or is such that $|\Lambda(t)| \to \infty$.

If Ω is bounded, then for sufficiently large λ, $\overline{\Omega}_\lambda = \emptyset$. However by the argument above, for each $(\lambda, w) \in \tilde{\mathfrak{B}}$, $\mathcal{R}(w) \cap \overline{\Omega}_\lambda \neq \emptyset$. Hence only the first possibility is allowed.

In the case where $(\lambda^*, 0)$ satisfies Theorem 3.13 and Ω is bounded, each of the bifurcating curves must form a closed loop. Either the two loops are the same, in which case they form [a set at least as complicated as] a figure of eight; or the two curves are distinct, in which case their union is [at least] triply connected. $\square$

Example 4.15. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\lambda, y) = \lambda^2 + \left(\frac{y+10}{10}\right)^2 + 2 - \frac{2}{10\lambda^4 + 10\left(y - \frac{1}{2}\right)^4 + 1},$$

The graph of f is approximately a paraboloid of revolution with minimum at approximately $(-10, 0)$, streched in the y direction, with a sharp indentation in the region $[-1.1, 1.1] \times [-1.0, 2.0]$. f attains its minimum value, (approximately) 1.1 at (approximately) $(0, 0.4)$, so is bounded below by 1.

We shall show that the side of the dip closer to the minimum of the "paraboloid", and the far side of the "paraboloid" satisfy the conditions given for Ω and Ω' respectively in Theorem 4.14.

We have

$$f^{(0,1)}(\lambda, y) = \frac{2(y+10)}{10} + \frac{80\left(y - \frac{1}{2}\right)^3}{\left(10\lambda^4 + 10\left(y - \frac{1}{2}\right)^4 + 1\right)^2},$$

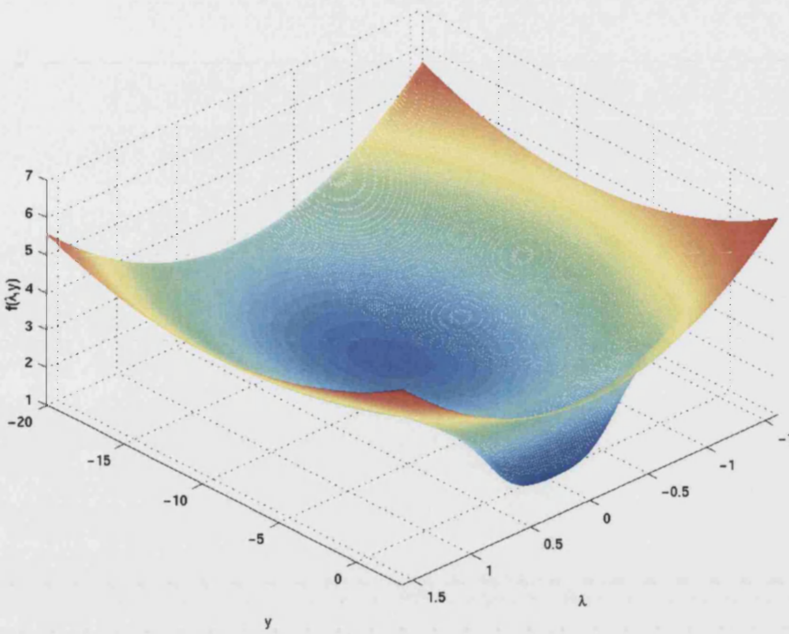

Figure 4.3: Plot of f for $(\lambda, y) \in [-1.6, 1.6] \times [-20, 3]$

so $f^{(0,1)}$ is negative on a set Ω' contained in $(-\infty, -9.9] \times \mathbb{R}$ and on a region Ω contained in $[-1,1] \times [-0.9, 0.4]$. For $\lambda \in [-1,1]$, $f(\lambda,0) \in [1.7, 3.8]$, so $\pi\sqrt{\frac{f(\lambda,0)-1}{2}} \in [1.9, 3.8]$, so for each $\lambda \in \mathbb{R}$ such that $\Omega_\lambda \neq \emptyset$, $h_\lambda \geq 9.9 - 0.0 > 3.8 \geq \pi\sqrt{\frac{f(\lambda,0)-1}{2}}$.

Now

$$q(\lambda) = -\frac{6400\lambda^8 + 2080\lambda^4 - 3031}{10(80\lambda^6 + 13\lambda^2 + 240\lambda^4 + 23)(80\lambda^4 + 13)}, \tag{4.11}$$

so $q(0) = 3031/2990$ whilst $q(\pm 1/2) = 1438/3825$. By inspection it is seen that if $\lambda \in (-1/2, 0)$ then $q'(\lambda) > 0$, and if $\lambda \in (0, 1/2)$ then $q'(\lambda) < 0$.

It follows that (1.4) has Crandall-Rabinowitz type bifurcations of solutions of fundamental period 2π in the intervals $(-1/2, 0)$ and $(0, 1/2)$, and since Ω is bounded, a loop of nonconstant solutions bifurcates from the line of trivial solutions at each of these. ■

Appendix A

Proof of Theorem 1.8

Choose a compact segment, $\tilde{E}$ of E°, and let $\mathcal{E} = \mathcal{E}(E, \tilde{E})$ and $\xi = \xi_{E,\tilde{E}}$ be as given in (2.10) and the remarks following. Note that $u = \xi u + (1 - \xi)u$.

(1) We shall show that $\mathcal{C}(\xi u) \in L^p_{2\pi}$ and that $\mathcal{C}((1-\xi)u)$ is smooth on $\tilde{E}$. First,

$$\int_{-\pi}^{\pi} |\xi(x)u(x)|^p \,\mathrm{d}x = \left\{ \int_E + \int_{S^1 \setminus E} \right\} |\xi(x)u(x)|^p \,\mathrm{d}x \leq \int_E |u(x)|^p < \infty,$$

since $\xi(x) = 0$ on $S^1 \setminus E$. Hence $\xi u \in L^p_{2\pi}$, and $\mathcal{C}(\xi u) \in L^p_{2\pi}$ by the theorem of M. Riesz (Theorem 1.7 (A)).

Next,

$$\mathcal{C}((1-\xi)u)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-\xi(y))u(y)}{\tan\frac{x-y}{2}} \mathrm{d}y = \frac{1}{2\pi} \int_{S^1 \setminus \mathcal{E}} \frac{(1-\xi(y))u(y)}{\tan\frac{x-y}{2}} \mathrm{d}y, \quad \text{(A.1)}$$

so $\mathcal{C}((1-\xi)u)$ is smooth on $\tilde{E}$, since for $y \in S^1 \setminus \mathcal{E}$, $x \mapsto \cot\frac{x-y}{2}$ is smooth on $\tilde{E}$.

It follows that $\mathcal{C}u = \mathcal{C}(\xi u) + \mathcal{C}((1-\xi)u) \in L^p(\tilde{E})$, as required.

(2) By (A.1), $\mathcal{C}((1-\xi)u)$ is smooth on $\tilde{E}$. We show that $\mathcal{C}(\xi u) \in C^{n,\alpha}_{2\pi}$. First by Leibnitz' rule,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\xi u) = \sum_{r=0}^{n} \binom{n}{r} \xi^{(r)} u^{(n-r)}.$$

If $x, t \in E$ or (without loss of generality) $x \in E$ and $t \in S^1 \setminus E$, then

$$\begin{aligned}
&|\xi^{(r)}(x)u^{(n-r)}(x) - \xi^{(r)}(t)u^{(n-r)}(t)| \\
&\quad \le |\xi^{(r)}(t)||u^{(n-r)}(x) - u^{(n-r)}(t)| + |\xi^{(r)}(x) - \xi^{(r)}(t)||u^{(n-r)}(x)|. \quad \text{(A.2)}
\end{aligned}$$

In either case, the second term on the right-hand side of (A.2) is bounded by $C|x-t|^\alpha$ for some constant C, independent of x and t (we denote any such constant by C in this proof), since ξ is smooth and, for $r \in \{0, \dots, n\}$, $u^{(n-r)}$ is bounded on E. If $t \in E$, then the first term on the right-hand side of (A.2) is bounded by $C|x-t|^\alpha$, since $u \in C^{n,\alpha}(E)$ and $\xi^{(r)}$ is bounded; otherwise it is zero, since ξ is zero on $S^1 \setminus E$.

If $x, t \in S^1 \setminus E$, then

$$|\xi^{(r)}(x)u^{(n-r)}(x) - \xi^{(r)}(t)u^{(n-r)}(t)| = 0.$$

It follows that $\xi u \in C^{n,\alpha}_{2\pi}$; so $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathcal{C}(\xi u) = \mathcal{C}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\xi u)\right) \in C^{0,\alpha}_{2\pi}$ by Privalov's theorem (Theorem 1.7 (B)).

It follows that $\mathcal{C}u = \mathcal{C}(\xi u) + \mathcal{C}((1-\xi)u) \in C^{n,\alpha}(\tilde{E})$, as required. $\square$

Appendix B

Bootstrapping Argument for Chapter 2

Lemma B.1. *(generalisation of lemma 3.3 in [4]) Suppose* $f \in C^0(J) \cap C^{0,\beta}(\hat{J})$ *where* J, $\hat{J}$ *and* β *are as in Theorem 2.1. If* $u \in W^{1,1}_{2\pi} \cap W^{1,p}_{\mathrm{loc}}(E)$ *with* $\mathcal{R}(u) \subset J$, $E \subset u^{-1}(\hat{J})$ *and* $p \in (1,\infty) \setminus \left\{\frac{\beta+1}{\beta}\right\}$, *then*

$$\mathcal{F}(u) \in \begin{cases} L_{\mathrm{loc}}^{\frac{p}{1-\beta(p-1)}}(u^{-1}(E^\circ)) & p < \frac{\beta+1}{\beta} \\ L^\infty_{\mathrm{loc}}(u^{-1}(E^\circ)) & p > \frac{\beta+1}{\beta} \end{cases}$$

Proof. Let $\tilde{E}$ be a compact segment of E°. Then for $x \in \tilde{E}$ we have that

$$\begin{aligned} \mathcal{F}(u)(x) &= \frac{1}{2\pi}\int_{x-\pi}^{x+\pi} \frac{f(u(x)) - f(u(y))}{\tan\frac{x-y}{2}} u'(y)\, dy \\ &= \frac{1}{2\pi}\left(\int_{\hat{E}} + \int_X\right) \frac{f(u(x)) - f(u(y))}{\tan\frac{x-y}{2}} u'(y)\, dy \\ &= \frac{1}{2\pi}(D(x) + S(x)), \end{aligned}$$

say, where $\hat{E} = \mathcal{E}(E, \tilde{E})$ (as given in (2.10)) and $X = S^1 \setminus \hat{E}$. Note that if $E = S^1$ then $\hat{E} = S^1$ also, so $D = \mathcal{F}(u)$ and $S \equiv 0$. We examine first the regularity of D.

Since $f \in C^{0,\beta}(\hat{J})$, $u \in W^{1,p}_{\mathrm{loc}}(E) \subset C^{0,1-\frac{1}{p}}_{\mathrm{loc}}(E)$ and $E \subset u^{-1}(\hat{J})$, there is a constant C independent of x and u (all such constants will be denoted C in this proof) such that for all $x \in \hat{E}$,

$$\begin{aligned} |D(x)| &\leq \frac{1}{2\pi} \int_{\hat{E}} \frac{|f(u(x)) - f(u(y))|}{|\tan \frac{x-y}{2}|} |u'(y)| \, \mathrm{d}y \\ &\leq C \int_{\hat{E}} \frac{|u(x) - u(y)|^{\beta}}{|x - y|} |u'(y)| \, \mathrm{d}y \\ &\leq C \|u'\|_{L^p(\hat{E})} \int_{\hat{E}} |x - y|^{\beta(1-\frac{1}{p})-1} |u'(y)| \, \mathrm{d}y, \end{aligned} \tag{B.1}$$

since for $y \in \hat{E} \subset (x - \pi, x + \pi)$, $|\tan \frac{x-y}{2}| \geq |\frac{x-y}{2}|$.

Now $u' \in L^p(\hat{E})$, so by the Hardy-Littlewood-Sobolev inequality (see (1.6)), the map $B : \hat{E} \to \mathbb{R}$ given by

$$B(x) = \int_{\hat{E}} |u'(y)| |x - y|^{\frac{\beta(p-1)-p}{p}} \, \mathrm{d}y$$

is in $L^r(\hat{E})$ with $\|B\|_{L^r(\hat{E})} \leq C \|u'\|_{L^p(\hat{E})}$, where

$$r = \begin{cases} \frac{p}{1-\beta(p-1)} & p < \frac{\beta}{\beta+1} \\ \infty & p > \frac{\beta}{\beta+1} \end{cases}$$

It follows from (B.1) that $D \in L^r(\hat{E})$ also, as required.

Now suppose $E \neq S^1$. For $x \in \tilde{E}$ and $y \in X$, $\left|\tan \frac{x-y}{2}\right| \geq \frac{1}{2}\mathrm{dist}(\tilde{E}, \partial\hat{E}) > 0$, so in all cases,

$$|S(x)| \leq \frac{4}{\mathrm{dist}(\tilde{E}, \partial\hat{E})} \|f\|_{L^\infty(\mathcal{R}(u))} \|u'\|_1.$$

Hence $S \in L^\infty(\tilde{E})$, as required. □

For completeness, we include here the following result, which covers the boundary case $\beta = 1$ and $u \in W^{1,1}_{2\pi} \cap W^{1,2}_{\mathrm{loc}}(E)$. In this case we also have that $\mathcal{F}(u) \in L^\infty_{\mathrm{loc}}(E^\circ)$, as follows.

By the argument at the end of the proof of Lemma B.1, $S \in L^\infty_{\mathrm{loc}}(E^\circ)$. We show that $D \in L^\infty_{\text{loc}}(E^\circ)$ also. Let $x \in \hat{E}$, and for $y \in \hat{E}$, let

$$h(x,y) = \frac{f(u(x)) - f(u(y))}{\tan\frac{x-y}{2}}. \tag{B.2}$$

We show that the map $y \mapsto h(x,y)$ is in $L^2(\hat{E})$ with norm bounded independently of x. We have that for some constant C, independent of x (we denote all such constants C here),

$$\int_{\hat{E}} h(x,y)^2 \, dy = \int_{\hat{E}} \frac{\{f(u(x)) - f(u(y))\}^2}{\tan^2\frac{x-y}{2}} \, dy \leq \|f\|^2_{C^{0,1}(\hat{J})} \int_{\hat{E}} \frac{|u(x) - u(y)|^2}{|x-y|^2} \, dy, \tag{B.3}$$

since $f \in C^{0,1}(\hat{J})$ and if $y \in (x - \pi, x + \pi)$ then $\left|\tan\frac{x-y}{2}\right| \geq \left|\frac{x-y}{2}\right|$. We now use Hardy's inequality (1.5) to dominate the integral on the right-hand side of (B.3)

$$\int_{\hat{E}} h(x,y)^2 \, dy \leq \|f\|^2_{C^{0,1}(\hat{J})} \int_{\hat{E}} \frac{\left\{\int_y^x u'(t) \, dt\right\}^2}{|x-y|^2} \, dy \leq C\|f\|^2_{C^{0,1}(\hat{J})} \|u'\|^2_{L^2(\hat{E})}. \tag{B.4}$$

It now follows from Hölder's inequality that $D \in L^\infty(\hat{E})$: for $x \in \hat{E}$, we have

$$|D(x)| \leq \int_{\hat{E}} |u'(y)||h(x,y)| \, dy \leq C\|f\|_{C^{0,1}(\hat{J})} \|u'\|^2_{L^2(\hat{E})}, \tag{B.5}$$

as required.

The following regularity result for solutions $w \in W^{1,1}_{2\pi} \cap W^{1,p}_{\text{loc}}(w^{-1}(\hat{J}) \backslash \mathcal{N}(f(w)))$ of (1.1) is a direct consequence of Lemma B.1.

Corollary B.2. *Let $f \in C^0(J) \cap C^{0,\beta}(\hat{J} \setminus \mathcal{N}(f))$, where J, $\hat{J}$ and β are as in Theorem 2.1, and $c \in \mathbb{R}$. If $w \in W^{1,1}_{2\pi} \cap W^{1,p}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ is a solution of (1.1) for some $p > 1$ then $w \in C^{0,\alpha}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ for each $\alpha \in (0,1)$.*

Proof. We show first that $w \in W^{1,q}_{\text{loc}}(w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w)))$ for some $q > \frac{\beta+1}{\beta}$.

If $p > \frac{\beta+1}{\beta}$ then there is nothing to prove. Suppose $p \leq \frac{\beta+1}{\beta}$. Let g :

$\left(1, \frac{\beta+1}{\beta}\right) \to \mathbb{R}$ be given by

$$g(q) = \frac{q}{1 - \beta(q-1)}$$

and let n be the least natural number such that $g^n(p) = g \circ \cdots \circ g(p) \geq \frac{\beta+1}{\beta}$ (such n exists, since on $\left(1, \frac{\beta+1}{\beta}\right)$, $g(q) > q$ and g is strictly convex). Let E_{3n+6} be a compact segment of $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ and let $E_1 \supset \cdots \supset E_{3n+6}$ be a string of nested compact segments of $w^{-1}(\hat{J}) \setminus \mathcal{N}(f(w))$ such that for $i \in \{2, \ldots, 3n+6\}$, $E_i \subset E_{i-1}^{\circ}$. We show first that $w \in W^{1,g^n(p)}(E_{3n})$.

Let, for $k \in \{1, \ldots, 3n+5\}$, $\xi_k = \xi_{E_k, E_{k+1}}$ be as given below (2.10)

By Lemma B.1, $\mathcal{F}(w) \in L^{g(p)}(E_1)$; so by (2.19), $\mathcal{C}w' \in L^{g(p)}(E_1)$ also, since $f(w)$ is bounded away from 0 on E_1. Now $\mathcal{C}w' = \mathcal{C}(\xi_1 w') + \mathcal{C}((1-\xi_1)w')$. By the proof of Theorem 1.8, $\mathcal{C}((1-\xi_1)w')$ is smooth on E_2. Hence $\mathcal{C}(\xi_1 w') \in L^{g(p)}(E_2)$. Now $w' \in L^p(E_1)$, so $\xi_1 w \in L^p_{2\pi}$. Hence by the theorem of M.Riesz, (Theorem 1.7 (A)), $\mathcal{C}(\xi_1 w') \in L^p_{2\pi} \subset L^1_{2\pi}$. It follows from Theorem 1.8 (1) that $\xi_1 w' \in L^{g(p)}(E_3)$. Hence $w \in W^{1,g(p)}(E_3)$, since on E_3, $\xi_1 w' \equiv w'$.

By an inductive argument it follows that $w \in W^{1,g^n(p)}(E_{3n})$. Since as $q \nearrow \frac{\beta+1}{\beta}$, $g(q) \to +\infty$, the same argument as above shows that if $g^n(p) = \frac{\beta+1}{\beta}$ then $w \in W^{1,q}(E_{3n+3})$ for some $q > \frac{\beta+1}{\beta}$.

It now follows (in all cases) from Lemma B.1 that $\mathcal{F}(w) \in L^\infty(E_{3n+4})$; and hence by (2.19) that $\mathcal{C}w' \in L^\infty(E_{3n+4})$. Hence, by the same argument as above, $w \in W^{1,p}(E_{3n+6})$ for each $p \in (1, \infty)$; so $w \in C^{0,\alpha}(E_{3n+6})$ for each $\alpha \in (0,1)$, as required. $\square$

Appendix C

Calculation of the Taylor Series of β About $(0,0)$ For Theorems 3.9 and 3.13

We calculate the third order Taylor polynomial about $(0,0)$ of the bifurcation equation (3.12) in terms of partial derivatives of f. We calculate first the partial Fréchet derivatives of F.

C.1 Calculation of partial Fréchet derivatives of F at $(\lambda, 0)$

Theorem C.1. *Let $m \in \mathbb{N} \cup \{0\}$ and suppose $f \in C^m(I \times J)$ where I and J are open intervals in $\mathbb{R}$ with $0 \in J$. Let $w \in \Upsilon$. Then $F(\cdot, w) : I \to L^2_{2\pi}$ (given by (3.1)) is m-times continuously differentiable on I, and for $\lambda \in I$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$,*

$$\partial^{m,0} F(\lambda, w)[\lambda_1, \ldots, \lambda_m]$$
$$= \left\{ f^{(m,0)}(\lambda, w)(1+\mathcal{C}w') + \mathcal{C}(f^{(m,0)}(\lambda, w)w') - f^{(m,0)}(\lambda, 0) \right\} \prod_{i=1}^{m} \lambda_i . \quad \text{(C.1)}$$

Proof. (C.1) holds for $m = 0$ by definition of F (3.1). Suppose that it holds for $m = k \in \mathbb{N} \cup \{0\}$ and let $f \in C^{k+1}(I \times J)$. Then for $\lambda \in I$, $w \in \Upsilon$ and $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ and λ_{k+1} in a neighbourhood of 0, we have that

$$\begin{aligned}
&\partial^{k,0} F(\lambda + \lambda_{k+1}, w)[\lambda_1, \dots, \lambda_k] - \partial^{k,0} F(\lambda, w)[\lambda_1, \dots, \lambda_k] \\
&= \left\{ \mathcal{C} \left(\left[f^{(k,0)}(\lambda + \lambda_{k+1}, w) - f(k,0)(\lambda, w) \right] w' \right) \right. \\
&\quad - \left[f^{(k,0)}(\lambda + \lambda_{k+1}, 0) - f^{(k,0)}(\lambda, 0) \right] \\
&\quad \left. + \left[f^{(k,0)}(\lambda + \lambda_{k+1}, w) - f^{(k,0)}(\lambda, w) \right] (1 + \mathcal{C} w') \right\} \prod_{i=1}^{k} \lambda_i.
\end{aligned}$$

Now by Taylor's theorem (1.19) we have that for $t \in S^1$,

$$f^{(k,0)}(\lambda + \lambda_{k+1}, w(t)) - f^{(k,0)}(\lambda, w(t)) = \lambda_{k+1} f^{(k+1,0)}(\lambda + \lambda_{k+1}, w(t)) + R_1(\lambda + \lambda_{k+1}, \lambda)(t)$$

where $R_1(\lambda + \lambda_{k+1}, \lambda)(t)$ is given by (1.20) as

$$R_1(\lambda + \lambda_{k+1}, \lambda)(t) = \lambda_{k+1} \int_0^1 f^{(k+1,0)}(\lambda + s\lambda_{k+1}) - f^{(k+1,0)}(\lambda)\ \mathrm{d}s \qquad \text{(C.2)}$$

We show that $\|R_1(\lambda + \lambda_{k+1}, \lambda)\|_\infty = o(|\lambda_{k+1}|)$. It will suffice to consider $\lambda_{k+1} \in [\lambda - \varepsilon, \lambda + \varepsilon]$ for some small $\varepsilon > 0$. Since $[\lambda - \varepsilon, \lambda + \varepsilon] \times \mathcal{R}(w)$ is compact, $f^{(k+1,0)}$ is uniformly continuous on this set. Hence as $\lambda_{k+1} \to 0$, the integral on the right-hand side of (C.2) converges uniformly to 0 as a function of t. It follows that as $\lambda_{k+1} \to 0$, $\|R_1(\lambda + \lambda_{k+1}, \lambda)\|_\infty / |\lambda_{k+1}| \to 0$, as required.

By the same argument, $f^{(k,0)}(\lambda + \lambda_{k+1}, 0) - f^{(k,0)}(\lambda, 0) = \lambda_{k+1} f^{(k+1,0)}(\lambda, 0) + S$ where $\|S\|_\infty / |\lambda_{k+1}| \to 0$ as $\lambda_{k+1} \to 0$.

It follows that

$$\begin{aligned}
&\partial^{k,0}F(\lambda+\lambda_{k+1},w)[\lambda_1,\dots,\lambda_k]-\partial^{k,0}F(\lambda,w)[\lambda_1,\dots,\lambda_k]\\
&\quad=\left\{\lambda_{k+1}f^{(k+1,0)}(\lambda,w)(1+\mathcal{C}w')\right.\\
&\qquad\left.+\,\mathcal{C}\left(\lambda_{k+1}f^{(k+1,0)}(\lambda,w)w'\right)-\lambda_{k+1}f^{(k+1,0)}(\lambda,0)\right\}\prod_{i=1}^{k}\lambda_i\\
&\qquad+\left\{R_1(\lambda+\lambda_{k+1},\lambda)(1+\mathcal{C}w')+\mathcal{C}(R_1(\lambda+\lambda_{k+1},\lambda)w')-S\right\}\prod_{i=1}^{k}\lambda_i.
\end{aligned} \tag{C.3}$$

By Hölder's inequality, all the terms on the last line of the right-hand side of (C.3) are $o(|\lambda_{k+1}|)$ in $L^2_{2\pi}$-norm as $\lambda_{k+1}\to 0$. Hence (C.1) holds for $m=k+1$ also; and hence for all $m\in\mathbb{N}\cup\{0\}$ by induction. The continuity of the Fréchet derivative of F follows from the continuity of the derivative of f. □

Theorem C.2. *Let $m\in\mathbb{N}\cup\{0\}$ and $n\in\mathbb{N}$, and suppose $f\in C^{m+n}(I\times J)$ where I and J are open intervals in $\mathbb{R}$ with $0\in J$. Then F (given by (3.1)) is $m+n$-times continuously differentiable on $I\times\Upsilon$ and for $\lambda\in I$, $w\in\Upsilon$, $\lambda_1,\dots,\lambda_m\in\mathbb{R}$ and $w_1,\dots w_n\in W^{1,2}_{2\pi}$,*

$$\begin{aligned}
&\partial^{m,n}F(\lambda,w)[\lambda_1,\dots,\lambda_m,w_1,\dots,w_n]\\
&\quad=\left\{f^{(m,n)}(\lambda,w)(1+\mathcal{C}w')\prod_{j=1}^{n}w_j+\mathcal{C}\left(f^{(m,n)}(\lambda,w)w'\prod_{j=1}^{n}w_j\right)\right.\\
&\qquad\left.+f^{(m,n-1)}(\lambda,w)\sum_{j=1}^{n}\mathcal{C}w_j'\prod^{l,j,n}w_l+\mathcal{C}\left(f^{(m,n-1)}(\lambda,w)\sum_{j=1}^{n}w_j'\prod^{l,j,n}w_l\right)\right\}\prod_{i=1}^{m}\lambda_i
\end{aligned} \tag{C.4}$$

where $\prod^{l,j,n}$ denotes a product over all $l\in\{1,\dots,n\}\setminus\{j\}$.

Proof. Note first that, similarly to the remark made at the start of the proof of Theorem 4.4, Υ is open in $W^{1,2}_{2\pi}$, so we may differentiate $F(\lambda,\cdot)$ on this set.

Let $f \in C^{m+1}(I \times J)$; and let $\lambda \in I$ and $w \in \Upsilon$. By Theorem C.1, $\partial^{m,0}F$ exists and is continuous on $I \times \Upsilon$. Let $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and let w_1 be in a neighbourhood of 0 in $W^{1,2}_{2\pi}$. Then

$$\begin{aligned}
&\partial^{m,0}F(\lambda, w + w_1)[\lambda_1, \dots, \lambda_m] - \partial^{m,0}F(\lambda, w)[\lambda_1, \dots, \lambda_m] \\
&= \left\{ \left\{ f^{(m,0)}(\lambda, w + w_1) - f^{(m,0)}(\lambda, w) \right\} (1 + \mathcal{C}w') \right. \\
&\quad + \mathcal{C}\left(\left\{ f^{(m,0)}(\lambda, w + w_1) - f^{(m,0)}(\lambda, w) \right\} w' \right) \\
&\quad \left. + f^{(m,0)}(\lambda, w + w_1)\mathcal{C}w' + \mathcal{C}\left(f^{(m,0)}(\lambda, w + w_1)w' \right) \right\} \prod_{i=1}^{m} \lambda_i.
\end{aligned}$$

By Taylor's theorem (1.19), we have that for $t \in S^1$,

$$f^{(m,0)}(\lambda, w(t) + w_1(t)) - f^{(m,0)}(\lambda, w(t)) = w_1(t) f^{(m,1)}(\lambda, w(t)) + R_1(w + w_1, w)(t),$$

where $R_1(w + w_1, w)(t)$ is given by (1.20) as

$$R_1(w + w_1, w)(t) = w_1(t) \int_0^1 f^{(m,1)}(\lambda, w(t) + s w_1(t)) - f^{(m,1)}(\lambda, w(t))\, \mathrm{d}s. \quad \text{(C.5)}$$

We show that as $\|w_1\|_{W^{1,2}_{2\pi}} \to 0$, $\|R_1(w + w_1, w)\|_\infty = o\left(\|w_1\|_{W^{1,2}_{2\pi}} \right)$. It will suffice to restrict ourselves to the case $\|w_1\|_\infty \leq \varepsilon$ for some small $\varepsilon > 0$. As $\|w_1\|_{W^{1,2}_{2\pi}} \to 0$, $\|w_1\|_\infty \to 0$; since $f^{(m,1)}(\lambda, \cdot)$ is uniformly continuous on $\{a + b | a \in \mathbb{R}(w), |b| \leq \varepsilon\}$, the integral on the right-hand side of (C.5) converges uniformly to 0 as $\|w_1\|_\infty \to 0$. It follows that as $\|w_1\|_{W^{1,2}_{2\pi}} \to 0$,

$$\frac{\|R_1(w + w_1, w)\|_\infty}{\|w_1\|_{W^{1,2}_{2\pi}}} \leq \frac{\|R_1(w + w_1, w)\|_\infty}{\|w_1\|_\infty} \to 0,$$

as required. Hence

$$
\begin{aligned}
&\partial^{m,0}F(\lambda, w+w_1)[\lambda_1,\ldots,\lambda_m] - \partial^{m,0}F(\lambda, w)[\lambda_1,\ldots,\lambda_m] \\
&\quad = \left\{w_1 f^{(m,1)}(\lambda, w)(1+\mathcal{C}w') + \mathcal{C}\left(w_1 f^{(m,1)}(\lambda, w)w'\right)\right. \\
&\qquad \left. + f^{(m,0)}(\lambda, w)\mathcal{C}w' + \mathcal{C}\left(f^{(m,0)}(\lambda, w)w'\right)\right\}\prod_{i=1}^{m}\lambda_i \\
&\qquad + \Big\{R_1(w+w_1, w)(1+\mathcal{C}w'+\mathcal{C}w_1') + \mathcal{C}(R_1(w+w_1, w)(w'+w_1')) \\
&\qquad + w_1 f^{(m,1)}(\lambda, w)\mathcal{C}w_1' + \mathcal{C}\left(w_1 f^{(m,1)}(\lambda, w)w_1'\right)\Big\}\prod_{i=1}^{m}\lambda_i. \qquad \text{(C.6)}
\end{aligned}
$$

By Hölder's inequality, all the terms in the second set of curly brackets on the right-hand side of (C.6) are $o\left(\|w_1\|_{W^{1,2}_{2\pi}}\right)$ in $L^2_{2\pi}$-norm as $\|w\|_{W^{1,2}_{2\pi}} \to 0$. Hence (C.4) holds if $n = 1$. Suppose now that (C.4) holds if $n = k \in \mathbb{N} \cup \{0\}$ and let $f \in C^{m+k+1}(I \times J)$, $w \in \Upsilon$, $w_1, \ldots, w_k \in W^{1,2}_{2\pi}$, w_{k+1} in a neighbourhood of 0 in $W^{1,2}_{2\pi}$, $\lambda \in I$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. We have that

$$
\begin{aligned}
&\partial^{m,k}F(\lambda, w+w_{k+1})[\lambda_1,\ldots,\lambda_m, w_1,\ldots,w_k] - \partial^{m,k}F(\lambda, w)[\lambda_1,\ldots,\lambda_m, w_1,\ldots,w_k] \\
&\quad = \left\{ f^{(m,k)}(\lambda, w+w_{k+1})\mathcal{C}w_{k+1}'\prod_{j=1}^{k} w_j + \mathcal{C}\left(f^{(m,k)}(\lambda, w+w_{k+1})w_{k+1}'\prod_{j=1}^{k} w_j\right)\right. \\
&\quad + \left[f^{(m,k)}(\lambda, w+w_{k+1}) - f^{(m,k)}(\lambda, w)\right](1+\mathcal{C}w')\prod_{j=1}^{k} w_j \\
&\quad + \mathcal{C}\left(\left[f^{(m,k)}(\lambda, w+w_{k+1}) - f^{(m,k)}(\lambda, w)\right] w'\prod_{j=1}^{k} w_j\right) \\
&\quad + \left[f^{(m,k-1)}(\lambda, w+w_{k+1}) - f^{(m,k-1)}(\lambda, w)\right]\sum_{j=1}^{k}\mathcal{C}w_j'\prod^{l,j,k} w_l \\
&\quad \left. + \mathcal{C}\left(\left[f^{(m,k-1)}(\lambda, w+w_{k+1}) - f^{(m,k-1)}(\lambda, w)\right]\sum_{j=1}^{k} w_j'\prod^{l,j,k} w_l\right)\right\}\prod_{i=1}^{m}\lambda_i
\end{aligned}
$$

Now by Taylor's theorem (1.18), we have that for $t \in S^1$

$$\begin{aligned}
&f^{(m,k)}(\lambda, w(t) + w_{k+1}(t)) - f^{(m,k)}(\lambda, w(t)) \\
&\qquad\qquad = w_{k+1}(t) f^{(m,k+1)}(w(t)) + R_1'(w + w_{k+1}, w)(t), \\
&f^{(m,k-1)}(\lambda, w(t) + w_{k+1}(t)) - f^{(m,k-1)}(\lambda, w(t)) \\
&\qquad\qquad = w_{k+1}(t) f^{(m,k)}(w(t)) + R_1''(w + w_{k+1}, w)(t),
\end{aligned}$$

where $R_1'(w+w_{k+1}, w)(t)$ and $R_1''(w+w_{k+1}, w)(t)$ are the first order Taylor series remainders of the $C^1(J)$-functions $f^{(m,k)}(\lambda, \cdot)$ and $f^{(m,k-1)}(\lambda, \cdot)$ respectively at $w_{k+1}(t)$ about $w(t)$, defined similarly to (C.5). Hence

$$\begin{aligned}
&\partial^{m,k} F(\lambda, w+w_{k+1})[\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_k] - \partial^{m,k} F(\lambda, w)[\lambda_1, \ldots, \lambda_m, w_1, \ldots, w_k] \\
&= \Bigg\{ \left[f^{(m,k)}(\lambda, w) + w_{k+1} f^{(m,k+1)}(\lambda, w) + R_1'(w + w_{k+1}, w)\right] \mathcal{C} w_{k+1}' \prod_{j=1}^{k} w_j \\
&\quad + \mathcal{C}\left(\left[f^{(m,k)}(\lambda, w) + w_{k+1} f^{(m,k+1)}(\lambda, w) + R_1'(w + w_{k+1}, w)\right] w_{k+1}' \prod_{j=1}^{k} w_j \right) \\
&\quad + \left[w_{k+1} f^{(m,k+1)}(\lambda, w) + R_1'(w + w_{k+1}, w)\right] (1 + \mathcal{C} w') \prod_{j=1}^{k} w_j \\
&\quad + \mathcal{C}\left(\left[w_{k+1} f^{(m,k+1)}(\lambda, w) + R_1'(w + w_{k+1}, w)\right] w' \prod_{j=1}^{k} w_j \right) \\
&\quad + \left[w_{k+1} f^{(m,k)}(\lambda, w) + R_1''(w + w_{k+1}, w)\right] \sum_{j=1}^{k} \mathcal{C} w_j' \prod^{l,j,k} w_l \\
&\quad + \mathcal{C}\left(\left[w_{k+1} f^{(m,k)}(\lambda, w) + R_1''(w + w_{k+1}, w)\right] \sum_{j=1}^{k} w_j' \prod^{l,j,k} w_l \right) \Bigg\} \prod_{i=1}^{m} \lambda_i
\end{aligned}$$

By arguments similar to the one below (C.5), $R_1'(w+w_{k+1}, w)(t) = o\left(\|w_{k+1}\|_{W^{1,2}_{2\pi}}\right)$ and $R_1''(w + w_{k+1}, w)(t) = o\left(\|w_{k+1}\|_{W^{1,2}_{2\pi}}\right)$. It follows, by an argument similar to the one below (C.6) that (C.4) holds if $n = k + 1$ also. Hence (C.4) holds for all $n \in \mathbb{N}$ by induction. The continuity of the Fréchet derivative of F follows from the continuity of the derivative of f. $\square$

Corollary C.3. *Let I and J be intervals in $\mathbb{R}$ with $0 \in J$.*

Let $m \in \mathbb{N} \cup \{0\}$. If $f \in C^m(I \times J)$ then for all $\lambda \in I$ and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, $\partial^{m,0} F(\lambda, 0)[\lambda_1, \dots, \lambda_m] = 0$.

Let $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$. If $f \in C^{m+n}(I \times J)$ then for all $\lambda \in I$, $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and $w_1, \dots, w_n \in W^{1,2}_{2\pi}$,

$$\partial^{m,n} F(\lambda, 0)[\lambda_1, \dots, \lambda_m, w_1, \dots, w_n]$$
$$= \left\{ f^{(m,n)}(\lambda, 0) \prod_{j=1}^{n} w_j + f^{(m,n-1)}(\lambda, 0) \left[\sum_{j=1}^{n} \mathcal{C} w_j' \prod^{l,j,n} w_l + \mathcal{C}\left(\sum_{j=1}^{n} w_j' \prod^{l,j,n} w_l \right) \right] \right\} \prod_{i=1}^{m} \lambda_i. \tag{C.7}$$

C.2 Calculation of partial Fréchet derivatives of γ

In order to calculate the Taylor series of β it is also necessary to compute partial derivatives of γ up to second order. We assume that $f \in C^3(I \times J)$ and $q^* \neq 0$; and we consider F as a map from $\Upsilon \cap Z_1$ into Z_0. By the Lyapunov-Schmidt reduction (Lemma 1.23), $\gamma(\lambda, 0) = 0$ for all λ, so $\partial^{1,0}\gamma(\lambda, 0)1 = 0 = \partial^{2,0}\gamma(\lambda, 0)[1, 1]$; also, $\partial^{0,1}\gamma(\lambda^*, 0) = 0$. In order to compute other partial Fréchet derivatives of γ we differentiate the equation

$$QF(\lambda^* + \mu, t\Phi_{q^*} + \gamma(\lambda^* + \mu, t\Phi_{q^*})) = 0,$$

where Q denotes projection onto $\mathcal{R}(\partial^{0,1}F(\lambda^*, 0))$ $(= \{w \in Z_0 |\, \langle w, \Phi_{q^*} \rangle_2 = 0\})$, which holds for all μ and t (see (1.28)). Let $\theta = (\lambda^* + \mu, t\Phi_{q^*} + \gamma(\lambda^* + \mu, t\Phi_{q^*}))$. Then

$$Q\partial^{0,1}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}] = 0$$

so that

$$Q(\partial^{0,2}F(\theta)[\Phi_{q^*}+\partial^{0,1}\gamma(\lambda^*+\mu,t\Phi_{q^*})\Phi_{q^*},\Phi_{q^*}+\partial^{0,1}\gamma(\lambda^*+\mu,t\Phi_{q^*})\Phi_{q^*}]$$
$$+\partial^{0,1}F(\theta)[\partial^{0,2}\gamma(\lambda^*+\mu,t\Phi_{q^*})[\Phi_q,\Phi_q]])=0.$$

Hence if we let $\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]=\sum_{n=0}^{\infty}\alpha_n\Phi_n$ then by (C.7)

$$\begin{aligned}
&Q(\partial^{0,2}F(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]+\partial^{0,1}F(\lambda^*,0)[\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]])\\
&\quad=Q(\Phi_{q^*}^2f^{(0,2)}(\lambda^*,0)+f^{(0,1)}(\lambda^*,0)\{2q^*\Phi_{q^*}^2+2\mathcal{C}(\Phi_{q^*}\Phi_{q^*}')\}\\
&\qquad+f^{(0,1)}(\lambda^*,0)\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]+2f(\lambda^*,0)\mathcal{C}((\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}])'))\\
&\quad=Q\Bigg(\frac{1}{2}(1+\Phi_{2q^*})f^{(0,2)}(\lambda^*,0)+f^{(0,1)}(\lambda^*,0)\{q^*(1+2\Phi_{2q^*})\}\\
&\qquad+f^{(0,1)}(\lambda^*,0)\sum_{n=0}^{\infty}\alpha_n\Phi_n+2f(\lambda^*,0)\sum_{n=0}^{\infty}n\alpha_n\Phi_n\Bigg)=0,
\end{aligned}$$

so that $\alpha_n=0$ for all $n\in\mathbb{N}\cup\{0\}\setminus\{0,q^*,2q^*\}$ and

$$\alpha_0 = \frac{f^{(0,1)}(\lambda^*,0)}{2f(\lambda^*,0)}-\frac{f^{(0,2)}(\lambda^*,0)}{2f^{(0,1)}(\lambda^*,0)}, \tag{C.8}$$
$$\alpha_{2q^*} = \frac{f^{(0,2)}(\lambda^*,0)}{2f^{(0,1)}(\lambda^*,0)}-\frac{f^{(0,1)}(\lambda^*,0)}{f(\lambda^*,0)}. \tag{C.9}$$

Since γ maps into $\mathcal{R}(\partial^{0,1}F(\lambda^*,0))$ we have $\alpha_{q^*}=\langle\Phi_{q^*},\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]\rangle_2=0$. Hence

$$\begin{aligned}
&\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]\\
&\quad=\frac{f^{(0,1)}(\lambda^*,0)}{2f(\lambda^*,0)}-\frac{f^{(0,2)}(\lambda^*,0)}{2f^{(0,1)}(\lambda^*,0)}+\left\{\frac{f^{(0,2)}(\lambda^*,0)}{2f^{(0,1)}(\lambda^*,0)}-\frac{f^{(0,1)}(\lambda^*,0)}{f(\lambda^*,0)}\right\}\Phi_{2q^*}.
\end{aligned}$$

Also

$$Q(\partial^{1,1}F(\theta)[\Phi_{q^*}+\partial^{0,1}\gamma(\lambda^*+\mu,t\Phi_{q^*})\Phi_{q^*},1]+\partial^{0,1}F(\theta)[\partial^{1,1}\gamma(\lambda^*+\mu,t\Phi_{q^*})[\Phi_{q^*},1]]$$
$$+\partial^{0,2}F(\theta)[\Phi_{q^*}+\partial^{0,1}\gamma(\lambda^*+\mu,t\Phi_{q^*})\Phi_{q^*},\partial^{1,0}\gamma(\lambda^*+\mu,t\Phi_{q^*})1])=0.$$

Hence if we let $\partial^{1,1}\gamma(\lambda^*,0)[\Phi_{q^*},1] = \sum_{n=0}^{\infty}\xi_n\Phi_n$ then by (C.7),

$$\begin{aligned}
&Q(\partial^{1,1}F(\lambda^*,0)[\Phi_{q^*},1] + \partial^{0,1}F(\lambda^*,0)[\partial^{1,1}\gamma(\lambda^*,0)[\Phi_{q^*},1]]) \\
&= Q\left(f^{(1,1)}(\lambda^*,0)\Phi_{q^*} + 2q^*f^{(1,0)}(\lambda^*,0)\Phi_{q^*} + \sum_{n=0}^{\infty}\xi_n\Phi_n(f^{(0,1)}(\lambda^*,0) + 2nf(\lambda^*,0))\right) \\
&= 0.
\end{aligned}$$

Since $f^{(0,1)}(\lambda^*,0) + 2nf(\lambda^*,0) \neq 0$ for $n \in \mathbb{N}\cup\{0\}\setminus\{q^*\}$ we have that $\xi_n = 0$; similarly to the remarks below (C.8) $\xi_{q^*} = 0$. Hence $\partial^{1,1}\gamma(\lambda^*,0)[\Phi_{q^*},1] = 0$.

C.3 Calculation of partial derivatives of β

By (3.12) we have

$$\beta(\mu,t) = \langle\Phi_{q^*}, F(\lambda^*+\mu, t\Phi_{q^*} + \gamma(\lambda^*+\mu, t\Phi_{q^*}))\rangle_2 .$$

we differentiate this with respect to t and μ to find the partial derivatives of β of order up to three at $(0,0)$. Again, we assume $f \in C^3(I\times J)$ and $q^* \neq 0$; and we consider F as a map from $\Upsilon\cap Z_1$ into Z_0. Note that because for all μ

$$\beta(\mu,0) = \langle\Phi_{q^*}, F(\lambda^*+\mu, \gamma(\lambda^*+\mu,0))\rangle_2 = \langle\Phi_{q^*}, F(\lambda^*+\mu,0)\rangle_2 = 0,$$

we have that $\beta^{(n,0)}(0,0) = 0$ for all n. Now for all μ and t we have

$$\beta^{(0,1)}(\mu,t) = \left\langle\Phi_{q^*}, \partial^{0,1}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}]\right\rangle_2,$$

so that

$$\begin{aligned}
&\beta^{(1,1)}(\mu,t) \\
&= \left\langle\Phi_{q^*}, \partial^{1,1}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}, 1]\right\rangle_2 \\
&\quad + \left\langle\Phi_{q^*}, \partial^{0,2}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{1,0}\gamma(\lambda^*+\mu, t\Phi_{q^*})1]\right\rangle_2 \\
&\quad + \left\langle\Phi_{q^*}, \partial^{0,1}F(\theta)[\partial^{1,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})[\Phi_{q^*},1]]\right\rangle_2
\end{aligned}$$

and

$$
\begin{aligned}
&\beta^{(0,2)}(\mu, t) \\
&\quad = \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}]\big\rangle_2 \\
&\qquad + \big\langle \Phi_{q^*}, \partial^{0,1} F(\theta)[\partial^{0,2}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, \Phi_{q^*}]]\big\rangle_2 .
\end{aligned}
$$

Hence we have the following:

$$
\begin{aligned}
&\beta^{(2,1)}(\mu, t) \\
&\quad = \big\langle \Phi_{q^*}, \partial^{2,1} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, 1, 1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{1,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, 1, \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{1,1} F(\theta)[\partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1], 1]\big\rangle_2 \\
&\qquad + \big\langle \Phi_{q^*}, \partial^{1,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1, 1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,3} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \\
&\qquad\qquad\qquad \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1, \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1], \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{2,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})[1, 1]]\big\rangle_2 \\
&\qquad + \big\langle \Phi_{q^*}, \partial^{1,1} F(\theta)[\partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1], 1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1], \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,1} F(\theta)[\partial^{2,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1, 1]]\big\rangle_2 , \\
&\beta^{(1,2)}(\mu, t) \\
&\quad = \big\langle \Phi_{q^*}, \partial^{1,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, 1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,3} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \\
&\qquad\qquad\qquad \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1], \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{1,1}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, 1]]\big\rangle_2 \\
&\qquad + \big\langle \Phi_{q^*}, \partial^{1,1} F(\theta)[\partial^{0,2}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, \Phi_{q^*}], 1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,2} F(\theta)[\partial^{0,2}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, \Phi_{q^*}], \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 \\
&\qquad\quad + \big\langle \Phi_{q^*}, \partial^{0,1} F(\theta)[\partial^{1,2}\gamma(\lambda^* + \mu, t\Phi_{q^*})[\Phi_{q^*}, \Phi_{q^*}, 1], \partial^{1,0}\gamma(\lambda^* + \mu, t\Phi_{q^*})1]\big\rangle_2 ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\beta^{(0,3)}(\mu,t)\\
&= \left\langle \Phi_{q^*}, \partial^{0,3}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*},\right.\\
&\qquad\qquad \left.\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}, \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,2}F(\theta)[\partial^{0,2}\gamma(\lambda^*+\mu, t\Phi_{q^*})[\Phi_{q^*},\Phi_{q^*}], \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,2}F(\theta)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}, \partial^{0,2}\gamma(\lambda^*+\mu, t\Phi_{q^*})[\Phi_{q^*},\Phi_{q^*}]]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,2}F(\theta)[\partial^{0,2}\gamma(\lambda^*+\mu, t\Phi_{q^*})[\Phi_{q^*},\Phi_{q^*}], \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*+\mu, t\Phi_{q^*})\Phi_{q^*}]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,1}F(\theta)[\partial^{0,3}\gamma(\lambda^*+\mu, t\Phi_{q^*})[\Phi_{q^*},\Phi_{q^*},\Phi_{q^*}]]\right\rangle_2 .
\end{aligned}
$$

These expressions simplify considerably if we take $(\mu, t) = (0,0)$. We use the results of Sections C.1 and C.2 and find that

$$
\begin{aligned}
\beta^{(0,1)}(0,0) &= \left\langle \Phi_{q^*}, \partial^{0,1}F(\lambda^*,0)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*,0)\Phi_{q^*}]\right\rangle_2 = 0,\\
\beta^{(1,1)}(0,0) &= \left\langle \Phi_{q^*}, \partial^{1,1}F(\lambda^*,0)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*,0)\Phi_{q^*}, 1]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,2}F(\lambda^*,0)[\partial^{1,0}\gamma(\lambda^*,0)1, \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*,0)\Phi_{q^*}]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,1}F(\lambda^*,0)[\partial^{1,1}\gamma(\lambda^*,0)[\Phi_{q^*},1]]\right\rangle_2\\
&= \left\langle \Phi_{q^*}, \partial^{1,1}F(\lambda^*,0)[\Phi_{q^*},1]\right\rangle_2\\
&= \left\langle \Phi_{q^*}, f^{(1,1)}(\lambda^*,0)\Phi_{q^*} + 2q^* f^{(1,0)}(\lambda^*,0)\Phi_{q^*}\right\rangle_2\\
&= \pi\left(f^{(1,1)}(\lambda^*,0) - \frac{f^{(0,1)}(\lambda^*,0)f^{(1,0)}(\lambda^*,0)}{f(\lambda^*,0)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\beta^{(0,2)}(0,0) &= \left\langle \Phi_{q^*}, \partial^{0,2}F(\lambda^*,0)[\Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*,0)\Phi_{q^*}, \Phi_{q^*} + \partial^{0,1}\gamma(\lambda^*,0)\Phi_{q^*}]\right\rangle_2\\
&\quad + \left\langle \Phi_{q^*}, \partial^{0,1}F(\lambda^*,0)[\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]]\right\rangle_2\\
&= \left\langle \Phi_{q^*}, \partial^{0,2}F(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}]\right\rangle_2\\
&= \left\langle \Phi_{q^*}, f^{(0,2)}(\lambda^*,0)\Phi_{q^*}^2 + f^{(0,1)}(\lambda^*,0)\{2q^*\Phi_{q^*}^2 + 2\mathcal{C}(\Phi_{q^*}\Phi'_{q^*})\}\right\rangle_2\\
&= \left\langle \Phi_{q^*}, \frac{1}{2}f^{(0,2)}(\lambda^*,0)(1+\Phi_{2q^*}) + q^* f^{(0,1)}(\lambda^*,0)(1+2\Phi_{2q^*})\right\rangle_2\\
&= 0.
\end{aligned}
$$

By similar arguments we find that

$$\begin{aligned}\beta^{(2,1)}(0,0) &= \left\langle \Phi_{q^*}, \partial^{2,1}F(\lambda^*,0)[\Phi_{q^*},1,1]\right\rangle_2 \\ &= \pi\left(f^{(2,1)}(\lambda^*,0) - \frac{f^{(2,0)}(\lambda^*,0)f^{(0,1)}(\lambda^*,0)}{f(\lambda^*,0)}\right),\end{aligned}$$

and

$$\begin{aligned}\beta^{(1,2)}(0,0) &= \left\langle \Phi_{q^*}, \partial^{1,2}F(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*},1]\right\rangle_2 \\ &\quad + \left\langle \Phi_{q^*}, \partial^{1,1}F(\lambda^*,0)[\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}],1]\right\rangle_2 \\ &= \left\langle \Phi_{q^*}, \Phi_{q^*}^2 f^{(1,2)}(\lambda^*,0) + f^{(1,1)}(\lambda^*,0)\{2q^*\Phi_{q^*}^2 + 2\mathcal{C}(\Phi_{q^*}\Phi'_{q^*})\}\right\rangle_2 \\ &\quad + \left\langle \Phi_{q^*}, \partial^{1,1}F(\lambda^*,0)[\alpha_0 + \alpha_{2q^*}\Phi_{2q^*},1]\right\rangle_2 \\ &= 0.\end{aligned}$$

(α_0 and α_{2q^*} are given by (C.8-C.9)). Finally,

$$\begin{aligned}&\beta^{(0,3)}(0,0) \\ &= \left\langle \Phi_{q^*}, \partial^{0,3}F(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*},\Phi_{q^*}]\right\rangle_2 \\ &\quad +3\left\langle \Phi_{q^*}, \partial^{0,2}F(\lambda^*,0)[\partial^{0,2}\gamma(\lambda^*,0)[\Phi_{q^*},\Phi_{q^*}],\Phi_{q^*}]\right\rangle_2 \\ &= \left\langle \Phi_{q^*}, \Phi_{q^*}^3 f^{(0,3)}(\lambda^*,0) + \{3q^*\Phi_{q^*}^3 + 3\mathcal{C}(\Phi_{q^*}^2\Phi'_{q^*})\}f^{(0,2)}(\lambda^*,0)\right\rangle_2 \\ &\quad +3\left\langle \Phi_{q^*}, \Phi_{q^*}(\alpha_0+\alpha_{2q^*}\Phi_{2q^*})f^{(0,2)}(\lambda^*,0)\right. \\ &\qquad +\{(\alpha_0+\alpha_{2q^*}\Phi_{2q^*})q^*\Phi_{q^*} + 2q^*\alpha_{2q^*}\Phi_{2q^*}\Phi_{q^*} \\ &\qquad \left. +\mathcal{C}(\alpha_0+\alpha_{2q^*}\Phi_{2q^*})\Phi'_{q^*} + \alpha_{2q^*}\Phi_{q^*}\Phi'_{2q^*})\}f^{(0,1)}(\lambda^*,0)\right\rangle_2 \\ &= \left\langle \Phi_{q^*}, \Phi_{q^*}^3\{f^{(0,3)}(\lambda^*,0) + 3q^*f^{(0,2)}(\lambda^*,0)\} + 3f^{(0,2)}(\lambda^*,0)\mathcal{C}(\Phi_{q^*}^2\Phi'_{q^*})\right\rangle_2 \\ &\quad +3\left\langle \Phi_{q^*}, \alpha_0\{f^{(0,2)}(\lambda^*,0) + 2q^*f^{(0,1)}(\lambda^*,0)\}\Phi_{q^*}\right. \\ &\qquad +\alpha_{2q^*}\{f^{(0,2)}(\lambda^*,0) + 3q^*f^{(0,1)}(\lambda^*,0)\}\Phi_{q^*}\Phi_{2q^*} \\ &\qquad \left. +\alpha_{2q^*}f^{(0,1)}(\lambda^*,0)\mathcal{C}(\Phi_{2q^*}\Phi'_{q^*} + \Phi_{q^*}\Phi'_{2q^*})\right\rangle_2.\end{aligned}$$

We now apply the triginometric identities $\sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$ and $\cos^3 x = \frac{1}{4}(\cos 3x + 3\cos x)$ which give

$$\cos^2 x \sin x = \tfrac{1}{4}(\sin x + \sin 3x), \quad \cos 2x \sin x = \tfrac{1}{2}(\sin 3x - \sin x),$$
$$\cos 2x \cos x = \tfrac{1}{2}(\cos 3x + \cos x),$$

and find that

$$\begin{aligned}
&\beta^{(0,3)}(0,0) \\
&= \frac{1}{4}\left\langle \Phi_{q^*}, \{f^{(0,3)} + 3q^* f^{(0,2)}\}(\Phi_{3q^*} + 3\Phi_{q^*}) + 3q^*(\Phi_{q^*} + \Phi_{3q^*})f^{(0,2)} \right\rangle_2 \\
&\quad + \frac{3}{2}\left\langle \Phi_{q^*}, 2\alpha_0\{f^{(0,2)} + 2q^* f^{(0,1)}\}\Phi_{q^*} \right. \\
&\qquad \left. + \alpha_{2q^*}\{f^{(0,2)} + 3q^* f^{(0,1)}\}(\Phi_{3q^*} + \Phi_{q^*}) + q^*\alpha_{2q^*}(3\Phi_{3q^*} + \Phi_{q^*})f^{(0,1)} \right\rangle_2 \\
&= 3\pi\left\{\frac{1}{4}f^{(0,3)} + \left(\alpha_0 + q^* + \frac{1}{2}\alpha_{2q^*}\right)f^{(0,2)} + 2q^*(\alpha_0 + \alpha_{2q^*})f^{(0,1)}\right\} \\
&= 3\pi\left\{\frac{1}{4}f^{(0,3)} - \frac{f^{(0,1)}f^{(0,2)}}{2f} + \frac{f^{(0,2)2}}{4f^{(0,1)}} + \frac{f^{(0,1)3}}{2f^2}\right\}
\end{aligned}$$

where f and its partial derivatives are evaluated at $(\lambda^*, 0)$.

C.4 Calculation of the Taylor series of β around $(0,0)$ in the case $q'(\lambda^*) = 0$

If $q'(\lambda^*) = 0$, then $\beta^{(1,1)}(0,0) = 0$ and so the Taylor series of β about $(0,0)$ is given by

$$
\begin{aligned}
&\beta(\mu,t) \\
&= \frac{1}{3!}\left(\begin{pmatrix}\begin{pmatrix}\beta^{(3,0)}\\ \beta^{(2,1)}\end{pmatrix}\cdot\begin{pmatrix}\mu\\ t\end{pmatrix} & \begin{pmatrix}\beta^{(2,1)}\\ \beta^{(1,2)}\end{pmatrix}\cdot\begin{pmatrix}\mu\\ t\end{pmatrix}\\ \begin{pmatrix}\beta^{(2,1)}\\ \beta^{(1,2)}\end{pmatrix}\cdot\begin{pmatrix}\mu\\ t\end{pmatrix} & \begin{pmatrix}\beta^{(1,2)}\\ \beta^{(0,3)}\end{pmatrix}\cdot\begin{pmatrix}\mu\\ t\end{pmatrix}\end{pmatrix}\begin{pmatrix}\mu\\ t\end{pmatrix}\right)\cdot\begin{pmatrix}\mu\\ t\end{pmatrix}+\rho \\
&= \frac{1}{6}\{3\mu^2 t\beta^{(2,1)} + t^3\beta^{(0,3)}\} + \rho \\
&= \frac{\pi t}{2}\left\{\mu^2\left(f^{(2,1)} - \frac{f^{(2,0)}f^{(0,1)}}{f}\right)\right. \\
&\quad \left. + t^2\left(\frac{1}{4}f^{(0,3)} - \frac{f^{(0,2)}f^{(0,1)}}{2f} + \frac{f^{(0,2)2}}{4f^{(0,1)}} + \frac{f^{(0,1)3}}{2f^2}\right) + \rho'\right\},
\end{aligned}
\tag{C.10}
$$

where partial derivatives of β are evaluated at $(0,0)$ and partial derivatives of β are evaluated at $(\lambda^*,0)$. ρ and $\rho' = \frac{2\rho}{\pi t}$ are sums of higher order terms. Note that because for all n, $\beta^{(n,0)}(0,0) = 0$, all terms in the Taylor series of β about $(0,0)$ have a factor t.

Bibliography

[1] A. AMBROSETTI, G. PRODI, *A Primer of Nonlinear Analysis*, Cambridge University Press, Cambridge, 1993.

[2] K.I. BABENKO, Some remarks on the theory of surface waves of finite amplitude, *Soviet Math. Doklady*, **35 (3)** (1987), 500-603. (See also *loc. cit.* 647-650).

[3] F. DE BRUNO, Note sur une nouvelle formule de calcul différentiel, *Quart. J. Math.* **1**, (1856), 359-360.

[4] B. BUFFONI, E.N. DANCER, J.F. TOLAND, The regularity and local bifurcation of Stokes waves, *Arch. Rational Mech. Anal.* **152 (3)** (2000), 207-240.

[5] B. BUFFONI, E.N. DANCER, J.F. TOLAND, The Sub-Harmonic bifurcation of Stokes Waves, *Arch. Rational Mech. Anal.* **152 (3)** (2000), 241-271.

[6] B. BUFFONI, J.F. TOLAND, *Analytic Theory of Global Bifurcation - An Introduction*, Princeton University Press, Princeton, 2003.

[7] G.B. FOLLAND, Remainder estimates in Taylor's theorem, *Amer. Math. Monthly* **97**, (1990), 233-235.

[8] L.E. FRAENKEL, Formulae for high derivatives of composite functions, *Math. Proc. Camb. Phil. Soc* **83**, (1978), 159-165.

[9] D. GILBARG, N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, 2nd edn., Springer Verlag, Berlin and Heidelberg, 1983.

[10] M.W. HIRSCH, *Differential Topology*, Springer Verlag, New York, 1976.

[11] H. KIELHÖFER, Multiple eigenvalue bifurcation for Fredholm operators, *J. Reine Angew. Math* **358** (1985), 104-124.

[12] P. KOOSIS, *Introduction to H_p Spaces*, 2nd edn., Cambridge University Press, Cambridge, 1999.

[13] H. LEWY, A note on harmonic functions and a hydrodynamical application, *Proc. Amer. Math. Soc.* **3**, (1952), 111-113.

[14] E.H. LIEB, M. LOSS, *Analysis*, Amer. Math. Society, Providence, 1997.

[15] R.S. PALAIS, Morse Theory on Hilbert Manifolds, *Topology* **2**, (1963) 299-340.

[16] W. RUDIN, *Real and Complex Analysis. 3rd edition*, McGraw-Hill, New York, 1986.

[17] E. SHARGORODSKY, J.F. TOLAND, A Riemann-Hilbert Problem and the Bernoulli Boundary Condition in the Variational Theory of Stokes Waves, *Annales Inst. H.Poincaré, Analyse Non Linéaire* **20 (1)** (2003), 37-52.

[18] E. SHARGORODSKY, J.F. TOLAND, Riemann-Hilbert Theory for Problems with Vanishing Coefficients that arise in Nonlinear Hydrodynamics, *J. Funct. Anal.* **197 (1)** (2003), 283-300.

[19] J.F. TOLAND, Stokes waves in Hardy spaces and as distributions, *J. Math. Pures Appl.* **79** (2000) 901-917.

[20] J.F. TOLAND, Stokes Waves, *Topological Methods in Nonlinear Analysis J. Juliusz Schauder Center* **7** (1996) 1-48.

[21] J.F. TOLAND, *personal communication*, 2001.

[22] ZYGMUND, *Trigonometric Series I & II,* Cambridge University Press, Cambridge, 1959.